A quick translation of made for the course in the fall '04

Tentamen i **Finansiella derivat och stokastisk analys** (CTH-TMA285) (GU-MAM695)
13 december 2003. Hjälpmedel: Beta.

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1. (3p) Find all (if any) real a, b, c, and d such that
$$W^3(t) + (at+b)W^2(t) + (ct+d)W(t)$$
will be a martingale.

2. (3p) Solve the following SDE for $0 \le t < T$ where $X(0) = 0$
$$dX(t) = \frac{X(t)}{T-t}\, dt + dW(t).$$
Let $V(t)$ denote the variance of $X(t)$. Compute $V(t)$ and show that $2V(\frac{s+t}{2}) \le V(s) + V(t)$ when $s, t \in (0, T)$.

3. (3p) Suppose the value of a stock is $S(t) = W(t)$ and the value for a bond $B(t) \equiv B(0) = 1$. Let us use the the financial strategy $(\phi(t), \psi(t)) = (2W(t), -t - W^2(t))$ for a portfolio which at time t has $\phi(t)$ stocks and $\psi(t)$ bonds. Show that the strategy is self-financing

4. (3p) A contract gives the buyer R kr at time τ, where
$$\tau = \inf_t [t > 0;\ W(t) \ge 1],$$
and where $W(t)$ is a normalized Wienerprocess. We will also assume that we have a constant interest rate r. What is the expected (with respect to the "real world" probability) value of this contract?

5. (3p) Show that the curve $(t, W(t))$, where $0 \le t \le 1$, has infinite length.

6. (4p) (Vasiček's model) Assume that the short interest rate, $r(t)$, satisfies
$$dr(t) = (b - a\, r(t))\, dt + \sigma\, dW(t), \quad 0 \le t \le T,$$
where a and b are positive constants, σ a constant volatilty and $(W(t))_{t \ge 0}$ a normalized real valued Wienerprocess w.r.t. the risk-neutral-measure Q. Decide the theoretical price at time $t = 0$ for a zero-coupon bound which gives the owner the amount 1 at time T.

7. (4p) (Feynman-Kacs formula) Suppose $u(x,t)$ solve the following parabolic pde.
$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma^2(t,x)}{2} \frac{\partial^2 u}{\partial x^2} + a(t,x) \frac{\partial u}{\partial x} + b(t,x) u(t,x) = 0,\ 0 \le t < T,\ x \in \mathbb{R} \\ u_{|t=T} = f \end{cases}$$
and suppose that $X(t)$ is a solution to the stochastic differential equation
$$dX(t) = a(t, X(t))\, dt + \sigma(t, X(t))\, dW(t),\ 0 \le t \le T.$$
Show (using Itô calculus) that under suitable regularity conditions (which?), the following holds:
$$u(0,x) = E[f(X(T)) e^{\int_0^T b(s, X(s))\, ds} | X(0) = x].$$

(Two forlulas on the back $\rightarrow$)

FORMULAS

- Suppose $X(t) = \alpha t + \sigma W(t),\ 0 \leq t \leq T$, where $(W(t))_{0\leq t\leq}$ is a real valued normalized Wienerprocess. $\alpha \in \mathbb{R}$ och $\sigma > 0$. Then

$$P\left[\max_{0\leq t\leq T} X(t) < x\right] = \Phi\left(\frac{x-\alpha T}{\sigma\sqrt{T}}\right) - e^{\frac{2\alpha x}{\sigma^2}}\,\Phi\left(-\frac{x+\alpha T}{\sigma\sqrt{T}}\right)$$

for all $x > 0$.

- If $a, b > 0$ then

$$\int_0^\infty \frac{1}{t^{3/2}\, e^{at+b/t}}\, dt = \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}}.$$