MATHEMATICS, Chalmers and Göteborg University
Tentamen i Financial Derivative and Stochastic Analysi(CTH-TMA285) (GU-MAM695)
August 2006, 8.30 to 12.30 at *Väg och vatten*.
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Allowable handbook: βeta. No calculators are allowed.

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1. (3p) Let a stock price be modeled in the following way.

$$S(t) = a \exp\big((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\big),$$

where $\sigma > 0$.

   (a) Compute the variance of $S(t)$.
   (b) Compute $\lim_{t\to\infty} S(t)$ for the cases $r > \frac{1}{2}\sigma^2$ and $r < \frac{1}{2}\sigma^2$.
   (c) Compute $\lim_{t\to\infty} \mathbb{E}\big(S(t)\big)$, and $\lim_{t\to\infty} \mathrm{Var}\big(S(t)\big)$.

2. (3p) Let $\mathbb{P}$ be the Lebesgue measure on $\Omega = [0,1]$. Find another measure $\tilde{\mathbb{P}}$ on Ω such that for all subsets $A \in \mathcal{B}[0,1]$ such that $\mathbb{P}(A) = 0$ implies $\tilde{\mathbb{P}}(A) = 0$ (i.e. $\tilde{\mathbb{P}}$ is *absolute continuous* with respect to $\mathbb{P}$), but such that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent. Remember to show that your suggested measure $\tilde{\mathbb{P}}$ is indeed a probability measure.

3. (3p) Let $\varphi(S(T))$ be the risk-neutral probability density function of the asset price $S(T)$ at maturity time T of an European call option.

   (a) Derive the price of the European call option, at time 0, for the strike price K at maturity time T, where we assume that the interest rate is constant r.
   (b) Let now c be the option price above, and let c_- and c_+ be the prices where the strike price is set to $K-\delta$ and $K+\delta$ respectively. We assume that δ is "small". Give an approximation of the value of $\varphi(K)$.

4. (3p) Let $\{Z_n\}_{n=0}^N$ be a discrete sequence adapted to the filtration $\mathcal{F}_n$. Let the sequence $\{S_n\}_{n=0}^N$ be such that $S_N := Z_N$ and

$$S_n := \max\big(Z_n, \mathbb{E}(S_{n+1}|\mathcal{F}_n)\big), \text{ for } n \le N-1.$$

(The sequence $\{S_n\}$ is called the Snell envelope of $\{Z_n\}$.)

   (a) Show that $\{S_n\}$ is a supermartingale.
   (b) Show that $\{S_n\}$ is the smallest supermartingale such that $S_n \ge Z_n$ for all $n \le N$, i.e. $\{S_n\}$ is *dominating* $\{Z_n\}$.

5. (3p) Let

$$dS_k(t) = \alpha_k(t)dt + \beta_k(t)\,dW(t),$$

for all $t \in [0,T]$ and $k = \{1,2\}$, where we also have that

$$\int_0^T |\alpha_k(t)|\,dt < \infty, \text{ and } \int_0^T \beta_k^2(t)\,dt < \infty.$$

Show that $dS_1 = dS_2$ if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

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6. (4p) (Change of measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}Z = 1$. For all $A \in \mathcal{F}$, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, dP(\omega).$$

Show that $\widetilde{\mathbb{P}}$ is a probability measure and that $\widetilde{\mathbb{E}}X = \mathbb{E}[XZ]$, for any non-negative random variable X.

7. (4p) (Central limit)
Let M_k be a symmetric random walk, i.e.

$$\mathbb{P}(M_{k+1} = M_k + 1) = \mathbb{P}(M_{k+1} = M_k - 1) = \frac{1}{2}.$$

Let us fix a positive integer n and let us for all t such that nt is a positive integer, define

$$W^{(n)}(t) = \frac{M_{nt}}{\sqrt{n}}.$$

Show that as $n \to \infty$, $W^{(n)}(t)$ converges to the normal distribution with mean zero and variance t.