MATHEMATICS, Chalmers and Göteborg University
Tentamen i Financial Derivative and Stochastic Analysi(CTH-TMA285) (GU-MAM695)
September 1 2006, at *Väg och vatten*.
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Allowable handbook: βeta. No calculators are allowed.

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1. (3p) Let a stock price be modeled in the following way.

$$S(t) = a \exp\Big((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big),$$

where $\sigma > 0$.

   (a) Compute the variance of $S(t)$.
   (b) Compute $\lim_{t\to\infty} S(t)$ for the cases $r > \frac{1}{2}\sigma^2$ and $r < \frac{1}{2}\sigma^2$.
   (c) Compute $\lim_{t\to\infty} \mathbb{E}\big(S(t)\big)$, and $\lim_{t\to\infty} \mathrm{Var}\big(S(t)\big)$.

Solution:

$$\mathrm{Var}\big(S(t)\big) = a^2 e^{2rt}\big(e^{\sigma^2 t} - 1\big).$$

2. (3p) Let $\mathbb{P}$ be the Lebesgue measure on $\Omega = [0, 1]$. Find another measure $\tilde{\mathbb{P}}$ on Ω such that for all subsets $A \in \mathcal{B}[0, 1]$ such that $\mathbb{P}(A) = 0$ implies $\tilde{\mathbb{P}}(A) = 0$ (i.e. $\tilde{\mathbb{P}}$ is *absolute continuous* with respect to $\mathbb{P}$), but such that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent. Remember to show that your suggested measure $\tilde{\mathbb{P}}$ is indeed a probability measure.

Solution: Get inspired by Exercise 1.10 on p. 45 in Shreve's book.

3. (3p) Let $\varphi(S(T))$ be the risk-neutral probability density function of the asset price $S(T)$ at maturity time T of an European call option.

   (a) Derive the price of the European call option, at time 0, for the strike price K at maturity time T, where we assume that the interest rate is constant r.
   (b) Let now c be the option price above, and let c_- and c_+ be the prices where the strike price is set to $K - \delta$ and $K + \delta$ respectively. We assume that δ is "small". Give an approximation of the value of $\varphi(K)$.

Solution: First we have that the price of the European option at time zero is

$$c = c(K) = e^{-rT} \int_{S(T)=K}^{\infty} (S(T) - K)\varphi(S(T))\, dS(T).$$

Now, differentiate twice with respect to the strike price K gives

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT}\varphi(K).$$

Hence we have that

$$\varphi(K) = e^{rT}\frac{\partial^2 c}{\partial K^2} \approx e^{rT}\frac{c_- - 2c + c_+}{\delta^2}.$$

4. (3p) Let $\{Z_n\}_{n=0}^N$ be a discrete sequence adapted to the filtration $\mathcal{F}_n$. Let the sequence $\{S_n\}_{n=0}^N$ be such that $S_N := Z_N$ and

$$S_n := \max\big(Z_n, \mathbb{E}(S_{n+1}|\mathcal{F}_n)\big), \text{ for } n \le N - 1.$$

(The sequence $\{S_n\}$ is called the Snell envelope of $\{Z_n\}$.)

(a) Show that $\{S_n\}$ is a supermartingale.

(b) Show that $\{S_n\}$ is the smallest supermartingale such that $S_n \geq Z_n$ for all $n \leq N$, i.e. $\{S_n\}$ is *dominating* $\{Z_n\}$.

Solution:

(a) $\{S_n\} \geq \mathbb{E}(S_{n+1}|\mathcal{F}_n)$ so $\{S_n\}$ is a supermartingale.

(b) Let $\{A_n\}$ be an arbitrary supermartingale dominating $\{Z_n\}$. We will show, by induction, that $\{A_n\}$ will also dominate $\{S_n\}$. First we note that since $S_N = Z_N$, that $A_N \geq S_N$. Let us now assume that $A_n \geq S_n$. Then

$$A_{n-1} \geq \mathbb{E}(A_n|\mathcal{F}_n) \geq \mathbb{E}(S_n|\mathcal{F}_{n-1}).$$

Furthermore, since $\{A_i\}$ dominates $\{Z_i\}$, we have especially that $A_{n-1} \geq Z_{n-1}$. Thus

$$A_{n-1} \geq \max\big(Z_{n-1}, \mathbb{E}(S_n|\mathcal{F}_{n-1})\big) = S_{n-1}.$$

Induction now gives us that $\{A_i\}$ dominates $\{S_i\}$ and we are done. (The Snell envelope i used in the study/evaluation of American options.)

5. (3p) Let

$$dS_k(t) = \alpha_k(t)dt + \beta_k(t)\,dW(t),$$

for all $t \in [0, T]$ and $k = \{1, 2\}$, where we also have that

$$\int_0^T |\alpha_k(t)|\,dt < \infty, \text{ and } \int_0^T \beta_k^2(t)\,dt < \infty.$$

Show that $dS_1 = dS_2$ if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Solution: One implication, i.e. $\Longleftarrow$, is trivial. We show the other one by letting $\alpha := \alpha_1 - \alpha_2$, $\beta := \beta_1 - \beta_2$, and

$$X(t) := \int_0^t \alpha(u)\,du + \int_0^t \beta(u)\,dW(u).$$

Then we have that

$$dX^2(t) = 2X(t)\,dX(t) + \beta^2(t)\,dt.$$

Hence

$$X^2(t) - X^2(0) = \int_0^t 2X(u)\,dX(u) + \int_0^t \beta^2(u)\,du. \qquad (1)$$

If $dS_1 = dS_2$ then

$$\int_0^t \alpha_1(u)\,du + \beta_1(u)\,dW(u) = \int_0^t \alpha_2(u)\,du + \beta_2(u)\,dW(u).$$

Then $X(t) = 0$ which implies, using (1), that $\beta(u) = 0$ for all $u \in [0, T]$. Then from the definition of $X(t)$ we see that

$$0 = X(t) = \int_0^t \alpha(u)\,du.$$

Hence $\alpha(u) = 0$ for all $u \in [0, T]$. Thus $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

6. (4p) (Change of measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}Z = 1$. For all $A \in \mathcal{F}$, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, dP(\omega).$$

Show that $\widetilde{\mathbb{P}}$ is a probability measure and that $\widetilde{\mathbb{E}}X = \mathbb{E}[XZ]$, for any non-negative random variable X.

Solution: See Theorem 1.6.1 on p. 33 in Shreve's book.

7. (4p) (Central limit)
Let M_k be a symmetric random walk, i.e.

$$\mathbb{P}(M_{k+1} = M_k + 1) = \mathbb{P}(M_{k+1} = M_k - 1) = \frac{1}{2}.$$

Let us fix a positive integer n and let us for all t such that nt is a positive integer, define

$$W^{(n)}(t) = \frac{M_{nt}}{\sqrt{n}}.$$

Show that as $n \to \infty$, $W^{(n)}(t)$ converges to the normal distribution with mean zero and variance t.

Solution: See Theorem 3.2.1 on page 89 in Shreve's.