MATHEMATICS, Chalmers and Göteborg University
Tentamen i Financial Derivative and Stochastic Analysi(CTH-TMA285) (GU-MAM695)
December 16 2005, 8.30 to 12.30 at *Väg och vatten*.
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Allowable handbook: βeta. No calculators are allowed.

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1. (3p) We will study a sequence of dice-tosses. Let D_i be the outcome of the toss number i, and let us define the random variable

$$X(n) := \sum_{i=1}^{n} D_i.$$

   (a) Compute

$$\mathbb{E}\big(X(n)|\mathcal{F}_{n-1}\big).$$

   (b) Construct a random variable $Z(n)$ based on $X(n)$ such that $Z(n) \to W(1)$ as $n \to \infty$, where $W(t)$ is the usual Brownian motion.

   (c) Compare your method for estimating $W(1)$ with the standard coin tossing method to generate an approximation of $W(1)$.

Solution:

   (a) $X(n-1) + 7/2$.

   (b)

$$Var(X(n)) = E(X^2) - (E(X))^2 = n\frac{1+4+9+25+36}{6} + n(n-1)(\frac{7}{2})^2 - (n\frac{7}{2})^2 = \frac{35n}{12}.$$

   Hence for large n

$$\frac{X(n) - \frac{21n}{6}}{\sqrt{\frac{35n}{12}}}$$

   is approximately $N(0,1)$ distributed.

2. (3p) (Vasiček's interest rate model)
Let

$$dR(t) = \big(\alpha - \beta R(t)\big)\, dt + \sigma\, dW(t),$$

where α, β, and σ are positive constants.

   (a) Compute and simplify $d(e^{\beta t}R(t))$ to a form not including $R(t)$.

   (b) Integrate the equation you got above and solve for $R(t)$.

   (c) Use the formula above to compute $\mathbb{E}[R(t)]$.

Solution:

   (a)

$$d(e^{\beta t}R(t)) = \ldots = e^{\beta t}\big(\alpha\, dt + \sigma\, dW(t)\big).$$

   (b)

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\big(1 - e^{-\beta t}\big) + \sigma e^{-\beta t}\int_0^t e^{\beta u}\, dW(u).$$

(c)

$$\mathbb{E}[R(t)] = \text{ An Itô–integral has expectation zero, since it is a martingale } = e^{-\beta t}R(0)+\frac{\alpha}{\beta}\left(1-e^{-\beta t}\right).$$

3. (3p) Suppose we have a model of market including two stocks with values $S_1(t)$ and $S_2(t)$ at time t. Let us also assume that the interest rate, r, is constant in this model. The two stocks are modeled in the standard way, i.e.

$$dS_1(t) = \alpha_1 S_1(t)\,dt + \sigma_1 S_1(t)\,dW_1(t),$$

$$dS_2(t) = \alpha_2 S_2(t)\,dt + \sigma_2 S_2(t)\,dW_2(t),$$

where α_i and σ_i are constants. Furthermore, in this model we also have that $\alpha_1 = \alpha_2$, $\sigma_2 = 2\sigma_1$ and $W_1(t) = W_2(t)$ for all $t \geq 0$.

   (a) Define risk–neutral probability measure.

   (b) Show that in the market model described above, there does not exist any risk–neutral probability measure.

   Solution:

   (a) See Definition 5.4.3 on p. 228 in Shreve's.

   (b) See Example 5.4.4 on p. 229. Using the method in this example we see that there exist a portfolio–strategy ($\Delta_1(t) = 1/(S_1(t)\sigma_1)$ and $\Delta_2(t) = -1/(S_2(t)2\sigma_1)$) to obtain an arbitrage. The First fundamental theorem of asset pricing (Thm 5.4.7 on p. 231) then tells us that there can not exist any risk–neutral probability measure.

4. (3p) Let $X(t) = at + bW(t)$, where a and b are non-zero constants and $W(t)$ a Brownian motion. Let f be a real valued function which is (at least) two times differentiable. For which such functions is $f(X(t))$ a martingale?

   Solution: Use Itô-Doeblin's formula on $d(f(X(t))$ to obtain

$$d(f(X(t)) = f'(X(t))\,dX(t)+\frac{1}{2}f''X(t))\,dX(t)\,dX(t) = f'(X(t))\big(a\,dt+b\,dW(t)\big)+\frac{1}{2}f''(X(t))b^2\,dt.$$

   Now, $f(X(t))$ is a martingale if and only if $d(f(X(t))$ does not have any dt–term. That is if and only if

$$af'(X(t)) + \frac{1}{2}f''(X(t))b^2 = 0.$$

   Let $y(\cdot) = f'(\cdot)$ which gives us the following first linear ODE

$$ay(x) + \frac{1}{2}b^2y'(x) = 0,$$

   which we solve by multiplying with the integrating factor $e^{\frac{2a}{b^2}x}$. Hence we have that

$$f'(x) = y = Ce^{-\frac{2a}{b^2}x},$$

   where we recall that both a and b are non-zero. Integrating once more with respect to x gives us finally that

$$f(x) = D - \frac{Cb^2}{2a}e^{-\frac{2a}{b^2}x},$$

   where C and D are arbitrarily constants.

5. (3p) Let $v(t, S(t))$ denote the price at time t of an up-and-out call with barrier B, strike price K, and maturity date T. Furthermore, suppose that the volatility σ and the interest rate r are constant. At a given time $t_0 \in (0, T]$, we assume that the underlying asset $S(t)$ is in $(0, B)$ and that the Greeks are known, i.e. $v_x(t_0, x_0) = \delta$, $v_t(t_0, x_0) = \theta$, and $v_{xx}(t_0, x_0) = \gamma$, where $x_0 = S(t_0)$.

(a) What is the value of the up-and-out call at time t_0?

(b) Suppose $t_0 = T$. Show that

$$\frac{\delta^2}{\gamma\theta} \geq 2\sigma^2$$

when $S(T) < K$ and

$$(\delta - 1)^2 r^2 \geq 2\gamma\sigma^2(Kr + \theta)$$

when $S(T) > K$. (In this case we naturally use the left derivative for θ at T.)

Solution:

(a) Apply Theorem 7.3.1 on p. 301 in Shreve's book

(b) When $t_0 = T$ we have that $v(t_0, x_0) = (x_0 - K)^+$. Depending on if $x_0 < K$ or $x_0 > K$, we get a polynomial equation of second degree in x_0. In order for this to have a real root, we get the inequalities given.

6. (4p) (Quadratic variation of an Itô–process)

(a) Define the quadratic variation of a function $f(t)$ defined for all $t \in [0, T]$.

(b) Let

$$X(t) = X(0) + \int_0^t \Delta(u)\, dW(u) + \int_0^t \Theta(u)\, du,$$

where $W(t)$ is a Brownian motion, $X(0)$ is nonrandom, and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes. Derive the quadratic variation of the process $X(t)$.

Solution: See pages, 101 and 143-144.

7. (4p) (Girsanov's Theorem) State and prove Girsanov's Theorem in one dimension. (You do not need to prove other Theorems or Lemmas which you might use in your proof as long as you clearly state the results you are using.)

Solution: See Theorem 5.2.3 on page 212-214.