FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[TMA285]$**&GU**$[MAM695]$**)**

December16, 2006, morning (4 hours), v
No aids.
Each problem is worth 3 points.

Solutions

1. (Black-Scholes model) Suppose $S(0) < B, T > 0$, and $M(T) = \max_{0\le u\le T} S(u)$. Find the price $\Pi_Y(0)$ at time zero of a barrier option of European type paying the amount $Y = 1_{[M(T)<B]}$ to its owner at time of maturity T.

Solution. We have

$$\Pi_Y(0) = e^{-rT}\tilde{E}\,[Y]$$

$$= e^{-rT}\tilde{P}\left[\max_{0\le u\le T} S(u) < B\right] = e^{-rT}\tilde{P}\left[\max_{0\le u\le T}\left\{(r-\frac{\sigma^2}{2})T + \sigma\tilde{W}(T)\right\} < \ln\frac{B}{S(0)}\right].$$

Thus by the given formula below

$$\Pi_Y(0) = e^{-rT}\left(N(\frac{\ln\frac{B}{S(0)} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) - e^{\frac{2(r-\frac{\sigma^2}{2})\ln\frac{B}{S(0)}}{\sigma^2}}N(-\frac{\ln\frac{B}{S(0)} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\right)$$

$$= e^{-rT}\left(N(\frac{\ln\frac{B}{S(0)} - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) - (\frac{B}{S(0)})^{(\frac{2r}{\sigma^2}-1)}N(-\frac{\ln\frac{B}{S(0)} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}})\right).$$

2. Find an adapted process $(\Gamma(t))_{0\le t\le T}$ such that

$$W^3(t) + W^2(t) - 3tW(t) - t = \int_0^t \Gamma(s)dW(s),\ 0\le t\le T.$$

Solution. Set $u(t,x) = x^3+x^2-3tx-t$. Then $u'_t = -3x-1$, $u'_x = 3x^2+2x-3t$, and $u''_{xx} = 6x+2$ and it follows that

$$u'_t + \frac{1}{2}u''_{xx} = 0.$$

Thus by the Itô-Doeblin formula

$$d(W^3(t)+W^2(t)-3tW(t)-t)=du(t,W(t))$$

$$=u'_t(t,W(t))dt+u'_x(t,W(t))dW(t)+\frac{1}{2}u''_{xx}(t,W(t))dt$$

$$=u'_x(t,W(t))dW(t)$$

and we get

$$\Gamma(t)=u'_x(t,W(t))=3W^2(t)+2W(t)-3t.$$

3. (Black-Scholes model with d stocks) Suppose $a_1,...,a_d\in\mathbf{R}$ and $K>0$ and consider a derivative of European type paying the amount

$$Y=(\sum_{i=1}^{d}a_iS_i(T)-K)^+$$

to its owner at time of maturity T. Show that

$$\Pi_Y(t)\geq(\sum_{i=1}^{d}a_iS_i(t)-K)^+$$

where $\Pi_Y(t)$ denotes the price of the derivative at time t.

Solution. Set $\tau=T-t$. We have

$$\Pi_Y(t)=e^{-r\tau}\tilde{E}\left[(\sum_{i=1}^{d}a_iS_i(T)-K)^+\mid\mathcal{F}(t)\right]$$

and since the function $f(x)=(x-K)^+$ is convex, the conditional Jensen inequality shows that

$$\Pi_Y(t)=e^{-r\tau}\tilde{E}\left[f(\sum_{i=1}^{d}a_iS_i(T))\mid\mathcal{F}(t)\right]$$

$$\geq e^{-r\tau}f(\tilde{E}\left[\sum_{i=1}^{d}a_iS_i(T)\mid\mathcal{F}(t)\right])$$

$$= e^{-r\tau} f(\sum_{i=1}^{d} a_i S_i(t) e^{r\tau}) = (\sum_{i=1}^{d} a_i S_i(t) - K e^{-r\tau})^+$$

$$\geq (\sum_{i=1}^{d} a_i S_i(t) - K)^+.$$

4. Show that the stochastic process

$$Z(t) = \exp(\sigma W(t) - \frac{\sigma^2}{2} t), \ t \geq 0$$

is a martingale.

5. (Black-Scholes model with d stocks and interest rate r; $\tilde{P}$ is the risk-neutral measure and $\tilde{W}$ denotes a d-dimensional $\tilde{P}$-Brownian motion)

a) Let $N = (N(t))_{0 \leq t \leq T}$ be a strictly positive price process. Show that there exists a volatility vector process $\nu(t) = (\nu_1(t), ..., \nu_d(t))$, $0 \leq t \leq T$, such that

$$dN(t) = rN(t)dt + N(t)\nu(t) \cdot d\tilde{W}(t).$$

b) Let $S = (S(t))_{0 \leq t \leq T}$ and $N = (N(t))_{0 \leq t \leq T}$ be strictly positive price processes with volatility vector processes $(\sigma(t))_{0 \leq t \leq T}$ and $(\nu(t))_{0 \leq t \leq T}$, respectively. Prove that

$$d\frac{S(t)}{N(t)} = \frac{S(t)}{N(t)}(\sigma(t) - \nu(t)) \cdot d\tilde{W}^{(N)}(t)$$

where $\tilde{W}^{(N)}(t) = \tilde{W}(t) - \int_0^t \nu(u)du$, $0 \leq t \leq T$.

c) Find a probability measure $\tilde{P}^{(N)}$ such that $\tilde{W}^{(N)}$ is a $\tilde{P}^{(N)}$-Brownian motion.

A formula

For any $T, \sigma, m > 0$, and $\alpha \in \mathbf{R}$,

$$P\left[\max_{0 \leq t \leq T}(\alpha t + \sigma W(t)) < m\right]$$

$$= N(\frac{m - \alpha T}{\sigma\sqrt{T}}) - e^{\frac{2\alpha m}{\sigma^2}} N(-\frac{m + \alpha T}{\sigma\sqrt{T}}).$$