SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[TMA285]$**&GU**$[MAM695]$**)**

April 13, 2007, morning (4 hours), v
No aids.
Each problem is worth 3 points.

1. Let W be a real-valued Wiener process. Show that the process $X(t) = \exp(\frac{t}{2})\cos(W(t))$, $t \geq 0$, is martingale.

Solution. Set $u(t,x) = \exp(\frac{t}{2})\cos(x)$. By the Itô-Doeblin formula

$$dX(t) = du(t,W(t)) = u'_t(t,W(t))dt + u'_x(t,W(t))dW(t) + \frac{1}{2}u''_{xx}(t,W(t))dt$$

$$= \frac{1}{2}\exp(\frac{t}{2})\cos(W(t))dt - \exp(\frac{t}{2})\sin(W(t))dW(t) - \frac{1}{2}\exp(\frac{t}{2})\cos(W(t))dt$$

$$= -\exp(\frac{t}{2})\sin(W(t))dW(t).$$

Hence $X(t) = 1 - \int_0^t \exp(\frac{s}{2})\sin(W(s))dW(s)$, $t \geq 0$, is a martingale.

2. Consider a capital market with a discount process $D(t) = e^{-t}$, $0 \leq t \leq T$, and a stock price process

$$S(t) = S(0)\exp(-\frac{1}{6}t^3 - \frac{a}{2}t^2 + (1 - \frac{a^2}{2})t + aW(t) + \int_0^t u dW(u)),\ 0 \leq t \leq T,$$

where a and T are strictly positive real numbers. Let

$$h_S(t) = \frac{1}{(t+a)S(t)}e^{W(t)+\frac{t}{2}}$$

and

$$h_B(t) = \frac{1}{B(t)}(e^{W(t)+\frac{t}{2}} - h_S(t)S(t))$$

where $B(t) = 1/D(t)$.

(a) Show that

$$h_S(t)dS(t) + h_B(t)dB(t) = d(e^{W(t)+\frac{t}{2}}).$$

(b) What is the price at time zero of a European derivative, which pays off $e^{W(T)+\frac{T}{2}}$ at time T?

(c) Show that

$$\lim_{a\to 0+} \int_0^T (h_S(t)S(t))^2 dt = +\infty.$$

Solution. (a) By the Itô-Doeblin formula

$$dS(t) = S(t)\left\{(-\frac{1}{2}t^2 - at + 1 - \frac{a^2}{2})dt + adW(t) + tdW(t) + \frac{1}{2}(a+t)^2 dt\right\}$$

$$= S(t)(dt + (t+a)dW(t))$$

and, clearly,

$$dB(t) = B(t)dt.$$

Moreover,

$$h_B(t) = \frac{1}{B(t)}(1 - \frac{1}{(t+a)})e^{W(t)+\frac{t}{2}}.$$

Hence

$$h_S(t)dS(t) + h_B(t)dB(t) =$$

$$\frac{1}{(t+a)S(t)}e^{W(t)+\frac{t}{2}}S(t)(dt + (t+a)dW(t))$$

$$+\frac{1}{B(t)}(1 - \frac{1}{(t+a)})e^{W(t)+\frac{t}{2}}B(t)dt$$

$$= e^{W(t)+\frac{t}{2}}dt + e^{W(t)+\frac{t}{2}}dW(t)$$

and

$$d(e^{W(t)+\frac{t}{2}}) = e^{W(t)+\frac{t}{2}}(dW(t) + \frac{1}{2}dt)$$

$$+\frac{1}{2}e^{W(t)+\frac{t}{2}}dt = e^{W(t)+\frac{t}{2}}dt + e^{W(t)+\frac{t}{2}}dW(t).$$

Consequently,

$$h_S(t)dS(t) + h_B(t)dB(t) = d(e^{W(t)+\frac{t}{2}}).$$

(b) $(e^{W(t)+\frac{t}{2}})_{|t=0} = 1$.

(c) We have

$$\int_0^T (h_S(t)S(t))^2 dt = \int_0^T \frac{1}{(t+a)^2} e^{2W(t)+t} dt \geq \exp(2 \min_{0 \leq t \leq T} W(t)) \int_0^T \frac{1}{(t+a)^2} dt$$

$$= \exp(2 \min_{0 \leq t \leq T} W(t))(\frac{1}{a} - \frac{1}{T+a}) \to \infty$$

as $a \to 0+$.

3. Let σ be a positive real number and u a non-negative real number.
(a) Suppose X is a real-valued random variable such as

$$E\left[e^{\lambda X}\right] \leq e^{\frac{\lambda^2 \sigma^2}{2}} \text{ if } \lambda \in \mathbf{R}.$$

Prove that

$$P\left[\mid X \mid \geq u\right] \leq 2e^{-\frac{u^2}{2\sigma^2}}.$$

(b) Suppose W is a real-valued Wiener process. If $(f(t))_{0 \leq t \leq 1}$ is an adapted process such that

$$\sup_{0 \leq t \leq 1,\ \omega \in \Omega} \mid f(t,\omega) \mid < \infty$$

the process $Z(t) = \exp(\int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds)$, $0 \leq t \leq 1$, is a martingale. Use this property to conclude that

$$P\left[\mid \int_0^t g(s)dW(s) \mid \geq u\right] \leq 2e^{-\frac{u^2}{2\sigma^2 t}}$$

if $0 < t \leq 1$ and $(g(t))_{0 \leq t \leq 1}$ is an adapted process such that

$$\sup_{0 \leq t \leq 1,\ \omega \in \Omega} \mid g(t,\omega) \mid \leq \sigma.$$

Solution. (a) It is enough to consider the special case $u > 0$. If $\alpha > 0$, then by the Markov inequality

$$P\left[X \geq u\right] = P\left[e^{\alpha X} \geq e^{\alpha u}\right] \leq e^{-\alpha u} E\left[e^{\alpha X}\right].$$

Hence

$$P[X \geq u] \leq e^{-\alpha u} e^{\frac{\alpha^2\sigma^2}{2}}$$

and since the quantity -$\alpha u + \frac{\alpha^2\sigma^2}{2}$ is minimal for $\alpha = \frac{u}{\sigma^2}$ we get

$$P[X \geq u] \leq e^{-\frac{u^2}{2\sigma^2}}.$$

In a similar way, if $\alpha > 0$,

$$P[X \leq -u] = P\left[e^{-\alpha X} \geq e^{\alpha u}\right] \leq e^{-\alpha u} E\left[e^{-\alpha X}\right]$$

$$\leq e^{-\alpha u} e^{\frac{\alpha^2\sigma^2}{2}} \leq e^{-\frac{u^2}{2\sigma^2}}.$$

Hence

$$P[\mid X \mid \geq u] = P[X \geq u] + P[X \leq -u] \leq 2e^{-\frac{u^2}{2\sigma^2}}.$$

(b) For every $\lambda \in \mathbf{R}$,

$$E\left[e^{\int_0^t \lambda g(s)dW(s)}\right] = E\left[e^{\int_0^t \lambda g(s)dW(s) - \frac{1}{2}\int_0^t (\lambda g)^2(s)ds} e^{\frac{\lambda^2}{2}\int_0^t g^2(s)ds}\right]$$

$$\leq E\left[e^{\int_0^t \lambda g(s)dW(s) - \frac{1}{2}\int_0^t (\lambda g)^2(s)ds} e^{\frac{\lambda^2\sigma^2 t}{2}}\right] = e^{\frac{\lambda^2\sigma^2 t}{2}} E\left[e^{\int_0^t \lambda g(s)dW(s) - \frac{1}{2}\int_0^t (\lambda g)^2(s)ds}\right]$$

$$= e^{\frac{\lambda^2\sigma^2 t}{2}}$$

since the process $\exp(\int_0^t \lambda g(s)dW(s) - \frac{1}{2}\int_0^t (\lambda g)^2(s)ds)$, $0 \leq t \leq 1$, is a martingale. Part (c) now follows from Part (a).

4. Let $T > 0$, let $\Pi = \{t_0, t_1, ..., t_n\}$ be a partition of $[0, T]$, and set

$$Q_\Pi = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2.$$

Show that $E[Q_\Pi] = T$ and Var(Q_Π)$\leq 2 \parallel \Pi \parallel T$.

5. The Vasicek model for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t), \ t \geq 0,$$

where α, β, and σ are positive constants and $R(0)$ is known. Find the distribution of $R(t)$.