FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH[$tma285$], GU[$MMA710$]), Period 2, autumn 2007

ASSIGNMENTS
Must be handed in at the latest Friday, November 30 at 9^{45}

Problem 1 Suppose $X : \Omega \to]0, \infty[$ is a bounded random variable. a) Show that

$$E[X \ln X] \geq E[X] \ln E[X].$$

b) Compute $E[X \ln X]$ and $E[X] \ln E[X]$ when X has a uniform distribution in the interval $]0, 1[$.

Problem 2 The random variable U has a uniform distribution in the interval $\left]-\frac{1}{2}, \frac{1}{2}\right[$. Set $X = \tan(\pi U)$. a) Find the cumulative distribution function of X. b) Find $\mu_X(\left[-1, \sqrt{3}/2\right])$.

Problem 3 Show that the process $e^{-\frac{t}{2}} \cosh W(t)$, $t \geq 0$, is a martingale.

Problem 4 Suppose $T > 0$ and $M(T) = \max_{0 \leq t \leq T} W(t)$. Find $E\left[e^{M(T)}\right]$.

Problem 5 Let $n \in \mathbf{N}_+$ and suppose the stochastic process $(V_n(t))_{0 \leq t \leq 1}$, has continuous sample paths, which are affine in each subinterval $\frac{k-1}{n} \leq t \leq \frac{k}{n}$, $k = 1, ..., n$. Moreover, assume $V_n(0) = 0$ and

$$V_n(\frac{k}{n}) = \frac{1}{n^\alpha} \sum_{j=1}^{k} X_j, \ k = 1, ..., n$$

where $(X_j)_{j=1}^n$ is an i.i.d. and $\alpha \in \mathbf{R}_+$. Use MATLAB to draw a picture of a realization of the process $(V_n(t))_{0 \leq t \leq 1}$, if $n = 1000$,
(title('Problem 5a')) $\alpha = 1/2$, and $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$.
(title('Problem 5b')) $\alpha = 1/2$, and $X_1 \in N(0, 1)$.
(title('Problem 5c')) $\alpha = 1$, and X_1 has the same distribution as

the random variable X in Problem 2.

Problem 6 (MATLAB; title('Problem 6'), legend('50 log-Brownian asset prices')) Consider a stock price process $(S(t))_{0\leq t\leq T}$, where

$$S(t) = S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W(t)}.$$

Plot 50 realizations of the stock price process if $S(0) = 1$, $\mu = 0.05$, $\sigma = 0.5$, and $T = 1$.

Problem 7 (MATLAB; title('Problem 7'), legend('Theoretic value','Actual value') Consider a stock price process $(S(t))_{0\leq t\leq T}$, where

$$S(t) = S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W(t)}.$$

Suppose $h = T/N$ and $t_n = nh$, $n = 0, ..., N$. Define $cash(t_0)$ by the equation

$$c(t_0, S(t_0), K, T) = \Delta_C(t_0)S(t_0) + cash(t_0)$$

where $\Delta_C(t) = c'_s(t, S(t), K, T)$. Set

$$\Pi_1 = \Delta_C(t_0)S(t_1) + cash(t_0)e^{rh}$$

where r is the interest rate. Next define $cash(t_1)$ by the equation

$$\Pi_1 = \Delta_C(t_1)S(t_1) + cash(t_1)$$

and

$$\Pi_2 = \Delta_C(t_1)S(t_2) + cash(t_1)e^{rh}$$

and continue the process until maturity T. Plot the theoretic call prices $c(t_i, S(t_i), K, T)$, $i = 0, ..., N$, and the portfolio values Π_i, $i = 0, ..., N$ in the same figure when $S(t_0) = 100$, $K = 105$, $T = 0.5$, $\mu = 0.05$, $r = 0.03$, $\sigma = 0.35$, and $N = 130$. Finally, find $\Pi_N - (S(T) - K)^+$.