SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH[$tma285$]**&GU**[$MMA710$]**)**

December 15, 2007, morning (4 hours), v
No aids.
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Each problem is worth 3 points.
REMARK: Below, if not otherwise stated, W denotes a one-dimensional Brownian motion.

1. Set

$$\Delta(t) = \int_0^t \sin(t+u)dW(u),\ 0 \leq t \leq 1$$

and

$$X = \int_0^1 \Delta(t)dW(t).$$

Find $E\left[\Delta^2(t))\right]$ and $E\left[X^2\right]$.

Solution. By the Itô isometry

$$E\left[\Delta^2(t)\right] = \int_0^t \sin^2(t+u)du$$

$$= \frac{1}{2}\int_0^t (1-\cos 2(t+u))du = \frac{1}{2}(t - \frac{1}{2}\sin 4t + \frac{1}{2}\sin 2t)$$

and

$$E\left[X^2\right] = E\left[\int_0^1 \Delta^2(t)dt\right] = \int_0^1 E\left[\Delta^2(t)\right]dt$$

$$= \int_0^1 \frac{1}{2}(t - \frac{1}{2}\sin 4t + \frac{1}{2}\sin 2t)dt = \frac{5}{16} + \frac{1}{16}\cos 4 - \frac{1}{8}\cos 2.$$

2. Let $W = (W_1(t), W_2(t))_{t\geq 0}$ be a standard Brownian motion in the plane and $(\mathcal{F}_t)_{t\geq 0}$ a filtration for W. Set

$$\begin{cases} X_1(t) = \int_0^t \cos(as)dW_1(s) \\ X_2(t) = \int_0^t \sin(as)dW_2(s) \end{cases}$$

where a is a real constant. Prove that the process

$$Y(t) = X_1^2(t) + X_2^2(t) - t, \ t \geq 0$$

is a martingale.

Solution. We have

$$\begin{cases} dX_1(t) = \cos(at)dW_1(t) \\ dX_2(t) = \sin(at)dW_2(t) \end{cases}$$

and

$$\begin{cases} (dX_1(t))^2 = \cos^2(at)dt \\ (dX_2(t))^2 = \sin^2(at)dt \end{cases}$$

Hence, by the Itô-Doeblin formula,

$$dY(t) = 2X_1(t)dX_1(t) + \frac{1}{2}2(dX_1(t))^2$$

$$+2X_2(t)dX_2(t) + \frac{1}{2}2(dX_2(t))^2 - dt$$

and we get

$$dY(t) = 2X_1(t)dX_1(t) + 2X_2(t)dX_2(t)$$

or

$$dY(t) = 2\cos(at)X_1(t)dW_1(t) + 2\sin(at)X_2(t)dW_2(t).$$

Now

$$Y(t) = 2\int_0^t \cos(au)X_1(u)dW_1(u) + 2\int_0^t \sin(au)X_2(u)dW_2(u), \ t \geq 0$$

is the sum of two martingales and it follows that $(Y(t))_{t\geq 0}$ is a martingale.

3. (Black-Scholes model for two stocks) Suppose $S_1(0) > S_2(0)$ and consider a derivative paying the amount K to its owner at time of maturity T if $S_1(t) > S_2(t)$ for all $t \in [0, T]$ and, otherwise, the payoff is zero. Find the price of the derivative at time zero.

Solution. Using standard notation,

$$S_i(t) = S_i(0)e^{(r-\frac{1}{2}|\sigma_i|^2)t+\sigma_i\cdot\tilde{W}(t)}$$

where $\sigma_i = [\sigma_{i1}\ \sigma_{i2}]$, $i = 1, 2$, and

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

is a volatility matrice.

Set $\alpha = \frac{1}{2}(\mid \sigma_1 \mid^2 - \mid \sigma_2 \mid^2)$ and $\delta = \sigma_2 - \sigma_1$ and note that $S_1(t) > S_2(t)$ for all $t \in [0, T]$ if and only if $\alpha t + \delta \cdot \tilde{W}(t) < \ln \frac{S_1(0)}{S_2(0)}$ for all $t \in [0, T]$. Therefore the price of the derivative at time zero equals

$$\Pi = e^{-rT} K\tilde{P} \left[\max_{0\leq t\leq T} (\alpha t + \delta \cdot \tilde{W}(t)) < \ln \frac{S_1(0)}{S_2(0)} \right]$$

where $\tilde{P}$ is the risk-neutral measure. Since $\tilde{W}$ is a standard Brownian motion under $\tilde{P}$ it follows that $(\delta \cdot \tilde{W}\ )/\mid \delta \mid$is a standard Brownian motion under $\tilde{P}$ and we get

$$\Pi = e^{-rT} K \left\{ N(\frac{\ln \frac{S_1(0)}{S_2(0)} - \alpha T}{\mid \delta \mid \sqrt{T}}) - (\frac{S_1(0)}{S_2(0)})^{\frac{2\alpha}{|\delta|^2}} N(-\frac{\ln \frac{S_1(0)}{S_2(0)} + \alpha T}{\mid \delta \mid \sqrt{T}}) \right\}.$$

4. Suppose $m > 0$ and $\tau_m = \min\{t \geq 0;\ W(t) = m\}$. Use the formula

$$P[\tau_m \leq t,\ W(t) \leq w] = P[W(t) \geq 2m - w],\ w \leq m,$$

to prove that τ_m has the density

$$f_{\tau_m}(t) = \frac{m}{t\sqrt{2\pi t}} e^{-\frac{m^2}{2t}},\ t > 0.$$

5. Suppose $W = (W_1(t), W_2(t))_{0\leq t\leq T}$ is a standard Brownian motion in the plane and $(\mathcal{F}_t)_{0\leq t\leq T}$ a filtration for W and consider the following market model with one stock. The stock price process is governed by the equation

$$dS(t) = \alpha S(t)dt + \sigma_1 S(t)dW_1(t) + \sigma_2 S(t)dW_2(t),\ 0 \leq t \leq T$$

where $\alpha, \sigma_1, \sigma_2 \in \mathbf{R}$ and $(\sigma_1, \sigma_2) \neq (0,0)$. Furthermore, it is assumed that the discount process is given by the equation $D(t) = e^{-rt}$, $0 \leq t \leq T$, where r is a positive constant. Find at least two risk-neutral measures.

Solution. The market price of risk equation reads

$$\alpha - r = \sigma_1 \theta_1(t) + \sigma_2 \theta_2(t),\ 0 \leq t \leq T.$$

First suppose $\sigma_2 \neq 0$. Since it is enough to find two risk-neutral measures we suppose $\gamma \in \mathbf{R}$ and set $\theta_1(t) = \gamma$ for all $0 \leq t \leq T$ and get

$$\theta_2(t) = \frac{\alpha - r - \sigma_1 \gamma}{\sigma_2},\ 0 \leq t \leq T.$$

Now $\tilde{P}_\gamma$ is a risk-neutral measure if

$$d\tilde{P}_\gamma = e^{-X_\gamma} dP$$

where

$$X_\gamma = \int_0^T (\gamma, \frac{\alpha - r - \sigma_1 \gamma}{\sigma_2}) \cdot dW(t) + \frac{1}{2} \int_0^T \mid (\gamma, \frac{\alpha - r - \sigma_1 \gamma}{\sigma_2}) \mid^2 dt$$

or

$$X_\gamma = \gamma W_1(T) + \frac{\alpha - r - \sigma_1 \gamma}{\sigma_2} W_2(T)$$

$$+ \frac{1}{2} \left\{ \gamma^2 + (\frac{\alpha - r - \sigma_1 \gamma}{\sigma_2})^2 \right\} T.$$

Since W_1 and W_2 are independent $X_{\gamma_1} \neq X_{\gamma_2}$ if $\gamma_1 \neq \gamma_2$. Thus $\tilde{P}_{\gamma_1} \neq \tilde{P}_{\gamma_2}$ if $\gamma_1 \neq \gamma_2$.

The case $\sigma_1 \neq 0$ can be treated in a similar way.

A formula

For any $T, \sigma, x > 0$, and $\alpha \in \mathbf{R}$

$$P\left[\max_{0 \leq t \leq T} (\alpha t + \sigma W(t)) < x \right]$$

$$= N(\frac{x - \alpha T}{\sigma \sqrt{T}}) - e^{\frac{2\alpha x}{\sigma^2}} N(-\frac{x + \alpha T}{\sigma \sqrt{T}}).$$