SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[tma285]$**&GU**$[MMA710]$**)**

March 28, 2008, morning (4 hours)
Each problem is worth 3 points. No aids.
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REMARK: Below, if not otherwise stated, W denotes a one-dimensional Brownian motion.

1. In the Bachelier model the money market account is a numéraire and the stock price is governed by the equation $dS(t) = \sigma S(0)dW(t), 0 \leq t \leq T$, where σ is a positive constant. Find (a) $P[S(T) > 0]$ (b) $P\left[\int_0^T S(t)dt > 0\right]$ (c) $E\left[(\int_0^T S(t)dt - K)^+\right]$, where K is a positive constant.

Solution. First note that $S(t) = S(0)(1 + \sigma W(t))$.

(a) Letting $G \in N(0,1)$, we have $P[S(T) > 0] = P\left[W(T) > -\frac{1}{\sigma}\right] = P\left[G < \frac{1}{\sigma\sqrt{T}}\right] = N(\frac{1}{\sigma\sqrt{T}})$.

(b) Clearly,

$$\int_0^T S(t)dt = TS(0) + \sigma S(0) \int_0^T W(t)dt.$$

Here

$$X = \int_0^T W(t)dt$$

is a Gaussian random variable and

$$E[X] = \int_0^T E[W(t)]\,dt = \int_0^T 0dt = 0.$$

Moreover,

$$E\left[X^2\right] = E\left[\int_0^T \int_0^T W(s)W(t)dsdt\right]$$

$$= \int_0^T \int_0^T E[W(s)W(t)]\,dsdt = \int_0^T \int_0^T \min(s,t)dsdt$$

$$= 2\int_0^T (\int_0^t s ds)dt = \frac{T^3}{3}$$

and it follows that $X \in N(0, \frac{T^3}{3})$. Hence, using the same notation as in the solution of Part (a),

$$P\left[\int_0^T S(t)dt > 0\right] = P\left[X > -\frac{T}{\sigma}\right]$$

$$= P\left[G < \frac{T}{\sigma\sqrt{\frac{T^3}{3}}}\right] = N(\frac{1}{\sigma}\sqrt{\frac{3}{T}}).$$

(c) Using the same notation as in the solutions of Parts (a) and (b),

$$E\left[(\int_0^T S(t)dt - K)^+\right] = E\left[(\sigma S(0)X - (K - TS(0)))^+\right]$$

$$= \sigma S(0)\frac{T^{3/2}}{\sqrt{3}}E\left[(G-L)^+\right]$$

where

$$L = \sqrt{3}\frac{K - TS(0)}{\sigma S(0)T^{3/2}}.$$

Moreover,

$$E\left[(G-L)^+\right] = \int_L^\infty (x-L)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{L^2}{2}} - LN(-L).$$

Thus

$$E\left[(\int_0^T S(t)dt - K)^+\right] = \sigma S(0)\frac{T^{3/2}}{\sqrt{3}}\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{L^2}{2}} - LN(-L)\right\}$$

with L as above.

2. (Black-Scholes model for two stocks) Suppose $T > 0$, $N \in \mathbf{N}_+$, $h = \frac{T}{N}$, and $t_n = nh$, $n = 0, ..., N$, and consider a derivative of European type paying

the amount $Y = \sum_{n=1}^{N} (\ln \frac{S_1(t_n)}{S_1(t_{n-1})} - \ln \frac{S_2(t_n)}{S_2(t_{n-1})})^2$ at time of maturity T. Find $\Pi_Y(0)$.

Solution. Suppose

$$S_i(t) = S_i(0) e^{(r-\frac{1}{2}|\sigma_i|^2)t+\sigma_i\cdot\tilde{W}(t)},\ i = 1, 2,$$

where $\tilde{W}$ is a standard 2-dimensional Brownian motion relative to the martingale measure $\tilde{P}$ and $\sigma_i = [\sigma_{i1}\ \sigma_{i2}] \neq 0$, $i = 1, 2$. Let

$$\varrho = \frac{\sigma_1 \cdot \sigma_2}{\mid \sigma_1 \mid\mid \sigma_2 \mid}$$

be the correlation.

Now

$$\begin{aligned}
&\ln \frac{S_1(t_n)}{S_1(t_{n-1})} - \ln \frac{S_2(t_n)}{S_2(t_{n-1})} \\
&= (r - \frac{1}{2} \mid \sigma_1 \mid^2)h + \sigma_1 \cdot (\tilde{W}(t_n) - \tilde{W}(t_{n-1})) \\
&\quad -(r - \frac{1}{2} \mid \sigma_2 \mid^2)h - \sigma_2 \cdot (\tilde{W}(t_n) - \tilde{W}(t_{n-1})) \\
&= \frac{1}{2}(\mid \sigma_2 \mid^2 - \mid \sigma_1 \mid^2)h + (\sigma_1 - \sigma_2) \cdot (\tilde{W}(t_n) - \tilde{W}(t_{n-1})).
\end{aligned}$$

Hence

$$\begin{aligned}
\Pi_Y(0) &= e^{-rT} \sum_{n=1}^{N} \tilde{E}\left[\left\{\frac{1}{2}(\mid \sigma_2 \mid^2 - \mid \sigma_1 \mid^2)h + (\sigma_1 - \sigma_2) \cdot (\tilde{W}(t_n) - \tilde{W}(t_{n-1}))\right\}^2\right] \\
&= e^{-rT}\left\{\frac{N}{4}(\mid \sigma_2 \mid^2 - \mid \sigma_1 \mid^2)^2 h^2 + \sum_{n=1}^{N} \tilde{E}\left[\left\{(\sigma_1 - \sigma_2) \cdot (\tilde{W}(t_n) - \tilde{W}(t_{n-1}))\right\}^2\right]\right\} \\
&= e^{-rT}\left\{\frac{N}{4}(\mid \sigma_2 \mid^2 - \mid \sigma_1 \mid^2)^2 h^2 + N \sum_{k=1}^{2} (\sigma_{1k} - \sigma_{2k})^2 h\right\} \\
&= e^{-rT} T \left\{\frac{1}{4}(\mid \sigma_2 \mid^2 - \mid \sigma_1 \mid^2)^2 h + \mid \sigma_1 \mid^2 - 2\varrho \mid \sigma_1 \mid^2 \mid \sigma_2 \mid^2 + \mid \sigma_2 \mid^2\right\}.
\end{aligned}$$

3. Let $W = (W_1, W_2)$ be a Brownian motion in the plane and suppose $A(t) = \int_0^t (-W_2(u))dW_1(u) + \int_0^t W_1(u)dW_2(u)$, $t \geq 0$, is the Lévy area process. Find $E\left[A^2(t)\right]$.

Solution. The Itô-Doeblin formula gives

$$dA^2(t) = 2A(t)dA(t) + \frac{1}{2}2(dA(t))^2.$$

Thus

$$dA^2(t) = 2A(t)((-W_2(t))dW_1(t) + W_1(t)dW_2(t))$$

$$+(W_2^2(t) + W_1^2(t))dt$$

and

$$A^2(t) = \int_0^t (-2A(u)W_2(u))dW_1(u) + \int_0^t 2A(u)W_1(u)dW_2(u)$$

$$+\int_0^t (W_2^2(u) + W_1^2(u))du.$$

Therefore

$$E\left[A^2(t)\right] = \int_0^t E\left[W_2^2(u) + W_1^2(u)\right] du = \int_0^t 2udu = t^2.$$

4. (a) State (but do not prove) the two-dimensional Itô-Doeblin formula. (b) Let $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ be Itô processes. Use Part (a) to show that $d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$.

5. Let $\Delta(t)$, $0 \leq t \leq 1$, be a nonrandom function of time such that $\int_0^1 \Delta^2(t)dt < \infty$ and define $I(t) = \int_0^t \Delta(s)dW(s)$, $0 \leq t \leq 1$. Prove that the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)ds$.