SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[tma285]$**&GU**$[MMA710]$)

August 26, 2008, morning (4 hours), M
Each problem is worth 3 points. No aids.
Examiner: Christer Borell, telephone number 0705292322

1. Let $W(t) = (W_1(t), W_2(t))$, $t \geq 0$, be a standard Brownian motion in the plane. Find

$$E\left[e^{W_1(1)-W_2(2)}\right].$$

Solution. Suppose $G \in N(0,1)$. Recall that

$$E\left[e^{\xi G}\right] = e^{\frac{\xi^2}{2}}.$$

We have

$$E\left[e^{W_1(1)-W_2(2)}\right] = E\left[e^{W_1(1)}e^{-W_2(2)}\right]$$
$$= \left\{ \begin{array}{c} W_1 \text{ and } W_2 \\ \text{are independent} \end{array} \right\}$$
$$= E\left[e^{W_1(1)}\right] E\left[e^{-W_2(2))}\right] = \left\{ \begin{array}{c} W_1(1) \in N(0,1) \\ W_2(2) \in N(0,2) \end{array} \right\}$$
$$= E\left[e^{G}\right] E\left[e^{\sqrt{2}G}\right] = e^{3/2}.$$

2. Consider a financial market with interest rate r and a stock price process $S = (S(t))_{t\geq 0}$ governed by a geometric Brownian motion with volatility $\sigma > 0$. The market prices a fixed European put with strike K and maturity T as in a Black-Scholes model with volatility $\frac{1}{2}\sigma$. Find an arbitrage.

(Hint: The market price $p(t, S(t))$ of the put satisfies the equation $p_t + \frac{\sigma^2 x^2}{8} p_{xx} + rxp_x - rp = 0$. Consider a portfolio with $X(0) = 0$. At each time t, the portfolio is long one European put, is short $p_x(t, S(t))$ shares of stock, and has the cash position $X(t) - p(t, S(t)) + S(t)p_x(t, S(t))$. You may use the well known fact that $p_{xx} > 0$ if $t < T$ without giving a proof.)

Solution. Consider a portfolio with $X(0) = 0$. At each time t, the portfolio is long one European put, is short $p_x(t, S(t))$ shares of stock, and has the cash position $X(t) - p(t, S(t)) + S(t)p_x(t, S(t))$. Now

$$dX(t) = dp(t, S(t)) - p_x(t, S(t))dS(t) + r\{X(t) - p(t, S(t)) + S(t)p_x(t, S(t))\}\,dt$$

$$= p_t(t, S(t))dt + p_x(t, S(t))dS(t) + \frac{\sigma^2 S^2(t)}{2} p_{xx}(t, S(t))dt$$

$$-p_x(t, S(t))dS(t) + r\{X(t) - p(t, S(t)) + S(t)p_x(t, S(t))\}\,dt$$

$$= \left\{ p_t(t, S(t)) + rS(t)p_x(t, S(t)) + \frac{(\frac{1}{2}\sigma)^2 S^2(t)}{2} p_{xx}(t, S(t)) - rp(t, S(t)) \right\} dt$$

$$+ \frac{\frac{3}{4}\sigma^2 S^2(t)}{2} p_{xx}(t, S(t))dt + rX(t)dt$$

$$= \frac{3\sigma^2 S(t)}{8} p_{xx}(t, S(t))dt + rX(t)dt.$$

Hence

$$d(X(t)e^{-rt}) = \frac{3\sigma^2 S(t)e^{-rt}}{8} p_{xx}(t, S(t))dt$$

and

$$X(t) = \frac{3\sigma^2 e^{rt}}{8} \int_0^t S(u)e^{-ru} p_{xx}(u, S(u))dt > 0 \text{ if } 0 < t \leq T.$$

3. (Black-Scholes model; two stocks) A derivative has the payoff

$$Y = \frac{1}{T} \int_0^T \sqrt{S_1(u)S_2(u)}du$$

at time of maturity T. Find the price $\Pi_Y(t)$ of the derivative at time t.

Solution. Using standard notation

$$S_1(t) = S_1(0)e^{(r - \frac{|\sigma_1|^2}{2})t + \sigma_1 \cdot \tilde{W}(t)}$$

and

$$S_2(t) = S_2(0)e^{(r - \frac{|\sigma_2|^2}{2})t + \sigma_2 \cdot \tilde{W}(t)}$$

where $\tilde{W}$ is a standard Brownian motion in the plane with respect to the martingale measure $\tilde{P}$. Hence, if $\tau = T - t$,

$$e^{r\tau}\Pi_Y(t) = \tilde{E}\left[\frac{1}{T}\int_0^T \sqrt{S_1(u)S_2(u)}du \mid \mathcal{F}_t\right]$$

$$= \tilde{E}\left[\frac{1}{T}\int_0^t \sqrt{S_1(u)S_2(u)}du + \frac{1}{T}\int_t^T \sqrt{S_1(u)S_2(u)}du \mid \mathcal{F}_t\right]$$

$$= \frac{1}{T}\int_0^t \sqrt{S_1(u)S_2(u)}du + \tilde{E}\left[\frac{1}{T}\int_t^T \sqrt{S_1(u)S_2(u)}du \mid \mathcal{F}_t\right].$$

$$= \frac{1}{T}\int_0^t \sqrt{S_1(u)S_2(u)}du + \frac{1}{T}\int_t^T \tilde{E}\left[\sqrt{S_1(u)S_2(u)} \mid \mathcal{F}_t\right] du.$$

Moreover, for any $u \geq t$,

$$\tilde{E}\left[\sqrt{S_1(u)S_2(u)} \mid \mathcal{F}_t\right]$$

$$= \sqrt{S_1(t)S_2(t)}\tilde{E}\left[\sqrt{e^{(r-\frac{|\sigma_1|^2}{2})(u-t)+\sigma_1\cdot(\tilde{W}(u)-\tilde{W}(t))}e^{(r-\frac{|\sigma_2|^2}{2})(u-t)+\sigma_2\cdot(\tilde{W}(u)-\tilde{W}(t))}} \mid \mathcal{F}_t\right]$$

$$= \sqrt{S_1(t)S_2(t)}\tilde{E}\left[\sqrt{e^{(2r-\frac{|\sigma_1|^2+|\sigma_2|^2}{2})(u-t)+(\sigma_1+\sigma_2)\cdot(\tilde{W}(u)-\tilde{W}(t))}} \mid \mathcal{F}_t\right]$$

$$= \sqrt{S_1(t)S_2(t)}\tilde{E}\left[e^{(r-\frac{|\sigma_1|^2+|\sigma_2|^2}{4})(u-t)+\frac{1}{2}(\sigma_1+\sigma_2)\cdot\tilde{W}(u-t)}\right]$$

$$= \sqrt{S_1(t)S_2(t)}e^{(r-\frac{|\sigma_1|^2+|\sigma_2|^2}{4})(u-t)+\frac{1}{8}|\sigma_1+\sigma_2|^2(u-t)}$$

$$= \sqrt{S_1(t)S_2(t)}e^{(r-\frac{1}{8}|\sigma_1-\sigma_2|^2)(u-t)}$$

and we get

$$\frac{1}{T}\int_t^T \tilde{E}\left[\sqrt{S_1(u)S_2(u)} \mid \mathcal{F}_t\right] du = \frac{1}{T}\sqrt{S_1(t)S_2(t)}\int_t^T e^{(r-\frac{1}{8}|\sigma_1-\sigma_2|^2)(u-t)}du$$

$$= \frac{1}{T(r-\frac{1}{8} \mid \sigma_1-\sigma_2 \mid^2)}\sqrt{S_1(t)S_2(t)}(e^{(r-\frac{1}{8}|\sigma_1-\sigma_2|^2)(T-t)} - 1)$$

(interpretted as $\frac{T-t}{T}\sqrt{S_1(t)S_2(t)}$ if $r = \frac{1}{8} \mid \sigma_1 - \sigma_2 \mid^2$). Now we conclude that $\Pi_Y(t)$ is equal to

$$\frac{e^{r\tau}}{T}\int_0^t \sqrt{S_1(u)S_2(u)}du + \frac{e^{-r\tau}}{T(r - \frac{1}{8} \mid \sigma_1 - \sigma_2 \mid^2)}\sqrt{S_1(t)S_2(t)}(e^{(r-\frac{1}{8}|\sigma_1-\sigma_2|^2)(T-t)} - 1).$$

4. Suppose W is a one-dimensional Brownian motion. Moreover, suppose $T \in]0, \infty[$ and let $\Pi = \{t_0, t_1, ..., t_n\}$ be a partition of the inteval $[0, T]$. Set

$$Q_\Pi = \sum_{j=0}^{n-1}(W(t_{j+1}) - W(t_j))^2$$

and show that $E[Q_\Pi] = T$ and Var$(Q_\Pi) \leq 2 \parallel \Pi \parallel T$.

5. Use the Itô-Doeblin formula to derivate the Black-Scholes-Merton partial differential equation for the price of a European call option.