**FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH[$tma285$], GU[$MMA710$]), Period 2, autumn 2008**

ASSIGNMENTS

Must be handed in at the latest Friday, November 28 at $15^{00}$

**Problem 0** Suppose $(X,Y)$ is a centred Gaussian random vector in the plane such that

$$\begin{cases} \sigma_X = \sqrt{E(X^2)} > 0 \\ \sigma_Y = \sqrt{E(Y^2)} > 0 \end{cases}$$

and let $\rho = \frac{E[XY]}{\sigma_X \sigma_Y}$. Find the conditional expectation $E\left[Y^2 \mid X\right]$.

**Problem 1** Suppose $X : \Omega \rightarrow ]0,\infty[$ is a bounded random variable. a) Show that

$$E\left[X \ln X\right] \geq E\left[X\right] \ln E\left[X\right].$$

b) Compute $E\left[X \ln X\right]$ and $E\left[X\right] \ln E\left[X\right]$ when $X$ has a uniform distribution in the interval $]0,1[$.

**Problem 2** The random variable $U$ has a uniform distribution in the interval $\left]-\frac{1}{2},\frac{1}{2}\right[$. Set $X = \tan(\pi U)$. a) Find the cumulative distribution function of $X$. b) Find $\mu_X(\left[-1,\sqrt{3}\right])$.

**Problem 3** Show that the process $e^{-\frac{t}{2}} \cosh W(t)$, $t \geq 0$, is a martingale.

**Problem 4** Suppose $T > 0$ and $M(T) = \max_{0 \leq t \leq T} W(t)$. Find $E\left[e^{M(T)}\right]$.

**Problem 5** Let $n \in \mathbf{N}_+$ and suppose the stochastic process $(V_n(t))_{0 \leq t \leq 1}$, has continuous sample paths, which are affine in each subinterval $\frac{k-1}{n} \leq t \leq \frac{k}{n}$, $k = 1,...,n$. Moreover, assume $V_n(0) = 0$ and

$$V_n(\frac{k}{n}) = \frac{1}{n^\alpha} \sum_{j=1}^{k} X_j,\ \ k = 1,...,n$$

where $(X_j)_{j=1}^n$ is an i.i.d. and $\alpha \in \mathbf{R}_+$. Use MATLAB to draw a picture of a realization of the process $(V_n(t))_{0 \leq t \leq 1}$, if $n = 1000$,

(title('Problem 5a')) $\alpha = 1/2$, and $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$.
(title('Problem 5b')) $\alpha = 1/2$, and $X_1 \in N(0, 1)$.
(title('Problem 5c')) $\alpha = 1$, and $X_1$ has the same distribution as the random variable $X$ in Problem 2.

**Problem 6** (MATLAB; title('Problem 6'), legend('50 log-Brownian asset prices')) Consider a stock price process $(S(t))_{0 \leq t \leq T}$, where

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

Plot 50 realizations of the stock price process if $S(0) = 1$, $\mu = 0.05$, $\sigma = 0.5$, and $T = 1$.

**Problem 7** (MATLAB; title('Problem 7'), legend('Theoretic value','Actual value') Consider a stock price process $(S(t))_{0 \leq t \leq T}$, where

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}.$$

Suppose $h = T/N$ and $t_n = nh$, $n = 0, ..., N$. Define $cash(t_0)$ by the equation

$$c(t_0, S(t_0), K, T) = \Delta_C(t_0)S(t_0) + cash(t_0)$$

where $\Delta_C(t) = c'_s(t, S(t), K, T)$. Set

$$\Pi_1 = \Delta_C(t_0)S(t_1) + cash(t_0)e^{rh}$$

where $r$ is the interest rate. Next define $cash(t_1)$ by the equation

$$\Pi_1 = \Delta_C(t_1)S(t_1) + cash(t_1)$$

and

$$\Pi_2 = \Delta_C(t_1)S(t_2) + cash(t_1)e^{rh}$$

and continue the process until maturity $T$. Plot the theoretic call prices $c(t_i, S(t_i), K, T)$, $i = 1, ..., N$, and the portfolio values $\Pi_i$, $i = 1, ..., N$ in the same figure when $S(t_0) = 100$, $K = 105$, $T = 0.5$, $\mu = 0.05$, $r = 0.03$, $\sigma = 0.35$, and $N = 130$. Finally, find $\Pi_N - (S(T) - K)^+$.