SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[tma285]$**&GU**$[MMA710]$**)**

December 16, 2008, morning (4 hours), v
Each problem is worth 3 points. No aids.
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If not otherwise stated W denotes a one-dimensional standard Brownian motion.

1. (Black-Scholes model) A portfolio consists of a shares of the stock and b shares of the bond, where $a, b > 0$ are constants. The value process of the portfolio is denoted by $N = (N(t))_{0 \leq t \leq T}$.

(a) Determine an adapted process $(\nu(t))_{0 \leq t \leq T}$ such that

$$dN(t) = N(t)(rdt + \nu(t)d\tilde{W}(t)), 0 \leq t \leq T.$$

(b) The process $\tilde{W}^{(N)}(t) = \tilde{W}(t) - \int_0^t \nu(u)du$, $0 \leq t \leq T$, is a standard Brownian motion with respect to the probability measure $\tilde{P}^{(N)}$. Determine adapted processes $(\gamma_S(t))_{0 \leq t \leq T}$ and $(\gamma_B(t))_{0 \leq t \leq T}$ such that

$$d\frac{S(t)}{N(t)} = \frac{S(t)}{N(t)}\gamma_S(t)d\tilde{W}^{(N)}(t)$$

and

$$d\frac{B(t)}{N(t)} = \frac{B(t)}{N(t)}\gamma_B(t)d\tilde{W}^{(N)}(t),$$

respectively.

Solution. (a) We have that

$$dN(t) = adS(t) + bdB(t)$$

$$= aS(t)(rdt + \sigma d\tilde{W}(t)) + brB(t)dt$$

$$= r(aS(t) + bB(t))dt + \sigma aS(t)d\tilde{W}(t)$$

$$= N(t)(rdt + \frac{\sigma aS(t)}{aS(t) + bB(t)}d\tilde{W}(t)).$$

Thus
$$\nu(t) = \frac{\sigma a S(t)}{aS(t) + bB(t)}.$$

(b) It is known that
$$\gamma_S(t) = \sigma - \nu(t)$$
$$= \sigma - \frac{\sigma a S(t)}{aS(t) + bB(t)} = \frac{\sigma b B(t)}{aS(t) + bB(t)}$$

and
$$\gamma_B(t) = 0 - \nu(t)$$
$$= -\frac{\sigma a S(t)}{aS(t) + bB(t)}.$$

2. (Black-Scholes model for two stocks) Suppose $t < T$. Find the price at time t of a derivative paying the amount $Y = \max(0, \ln \frac{S_1(T)}{S_2(T)})$ at time of maturity T.

Solution. Suppose
$$S_i(t) = S_i(0)e^{(r - \frac{|\sigma_i|^2}{2})t + \sigma_i \cdot \tilde{W}(t)}, \ i = 1, 2.$$

If $\sigma = \sigma_1 - \sigma_2$, $\tau = T - t$, and $x = \frac{S_1(t)}{S_2(t)}$, then
$$\frac{S_1(T)}{S_2(T)} = xe^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + \sigma \cdot (\tilde{W}(T) - \tilde{W}(t))}.$$

Now set $a = \frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2}$ and suppose $G \in N^{\tilde{P}}(0, 1)$ so that
$$\Pi_Y(t) = e^{-r\tau}\tilde{E}\left[(\ln \frac{S_1(T)}{S_2(T)})^+ \mid \mathcal{F}_t\right]$$
$$= e^{-r\tau}\tilde{E}\left[(\ln(xe^{a\tau + \sigma \cdot (\tilde{W}(T) - \tilde{W}(t))}))^+ \mid \mathcal{F}_t\right]$$
$$= e^{-r\tau}\tilde{E}\left[(\ln(xe^{a\tau + |\sigma|\sqrt{\tau}G}))^+\right]$$
$$= e^{-r\tau}\tilde{E}\left[(a\tau + \ln x - \mid \sigma \mid \sqrt{\tau}G)^+\right]$$
$$= e^{-r\tau}\int_{-\infty}^{\frac{a\tau + \ln x}{|\sigma|\sqrt{\tau}}} (a\tau + \ln x - \mid \sigma \mid \sqrt{\tau}y)e^{-\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}$$

$$= e^{-r\tau}\left\{(a\tau+\ln x)N(\frac{\ln x + a\tau}{\mid\sigma\mid\sqrt{\tau}}) + \mid\sigma\mid\sqrt{\frac{\tau}{2\pi}}e^{-\frac{1}{2}\left\{\frac{a\tau+\ln x}{|\sigma|\sqrt{\tau}}\right\}^2}\right\}.$$

3. Suppose T is a positive constant and $\sigma(t)$, $0 \le t \le T$, a nonrandom continuous function not identically equal to zero.

(a) Set

$$X(t) = \int_0^t \sigma(u)dW(u),\ 0 \le t \le T,$$

and

$$Y(t) = W(\int_0^t \sigma^2(u)du),\ 0 \le t \le T.$$

Prove that the processes $(X(t))_{0\le t\le T}$ and $(Y(t))_{0\le t\le T}$ have the same distribution.

(b) Consider an interest rate process $(R(t))_{0\le t\le T}$ with $R(0) = r$ and such that

$$dR(t) = (a - bR(t))dt + \sigma(t)dW(t),\ 0 \le t \le T,$$

where a and b are positive constants. Find the density of the random variable

$$Z = \max_{0\le t\le T}\left\{e^{bt}(R(t) - E\left[R(t)\right])\right\}.$$

Solution. (a). Note that $(X(t))_{0\le t\le T}$ ard $(Y(t))_{0\le t\le T}$ are mean-zero Gaussian processes. Therefore, if these processes have the same covariances they must have the same distribution.

If $0 \le s \le t \le T$,

$$E\left[X(s)X(t)\right] = E\left[\int_0^s \sigma(u)dW(u)\int_0^t \sigma(u)dW(u)\right]$$

$$= E\left[\int_0^t 1_{[0,s,]}(u)\sigma(u)dW(u)\int_0^t \sigma(u)dW(u)\right] = \int_0^t \sigma^2(u)1_{[0,s]}(u)du$$

$$= \int_0^s \sigma^2(u)du$$

and, therefore, for all $0 \le s, t \le T$,

$$E\left[X(s)X(t)\right] = \int_0^{\min(s,t)} \sigma^2(u)du.$$

Moreover,

$$E\left[Y(s)Y(t)\right] = E\left[W(\int_0^s \sigma^2(u)du)W(\int_0^t \sigma^2(u)du)\right]$$

$$= \min(\int_0^s \sigma^2(u)du, \int_0^t \sigma^2(u)du) = \int_0^{\min(s,t)} \sigma^2(u)du$$

and Part (a) is solved.

(b) We have

$$d(e^{bt}R(t)) = ae^{bt} + \sigma(t)e^{bt}dW(t)$$

and, hence,

$$e^{bt}R(t) = r + \frac{a}{b}(e^{bt} - 1) + \int_0^t \sigma(u)e^{bu}dW(u).$$

Now $e^{bt}E\left[R(t)\right] = E\left[e^{bt}R(t)\right] = r + \frac{a}{b}(e^{bt} - 1)$ and we get

$$Z = \max_{0 \le t \le T} \int_0^t \sigma(u)e^{bu}dW(u).$$

Next let x be a positive number. By using Part (a) it follows that

$$P\left[Z \ge x\right] = P\left[\max_{0 \le t \le T} W(\int_0^t \sigma^2(s)e^{2bs}ds) \ge x\right]$$

and the Bachelier double law gives

$$P\left[Z \ge x\right] = 2P\left[W(\int_0^T \sigma^2(u)e^{2bu}ds) \ge x\right]$$

$$= 2(1 - N(\frac{x}{a}))$$

where

$$a = \sqrt{\int_0^T \sigma^2(u)e^{2bu}ds}.$$

Hence, the density function of Z equals

$$f_Z(x) = \begin{cases} \frac{2}{a\sqrt{2\pi}} e^{-\frac{x^2}{2a^2}}, \ x > 0 \\ 0, \ \text{otherwise.} \end{cases}$$

4. Let $(\mathcal{F}(t))_{t\geq 0}$ be a filtration for W. Show that W is a Markov process.

5. Consider a model with m stocks and a discount process $(D(t))_{0\leq t\leq T}$. The stock price processes are Itô processes defined by a d-dimensional Brownian motion W and a filtration $\mathcal{F}_t$, $0 \leq t \leq T$ for W and, in addition,

$$D(t) = e^{-\int_0^t R(u)du}, 0 \leq t \leq T,$$

where the process $(R(t))_{0\leq t\leq T}$ is adapted.

(a) Explain the notion of risk-neutral measure.

(b) Let $X = (X(t))_{0\leq t\leq T}$ be a portfolio value process and $\tilde{P}$ a risk-neutral measure. Show that $(D(t)X(t))_{0\leq t\leq T}$ is a martingale under $\tilde{P}$.