SOLUTIONS: FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS (CTH$[tma285]$**&GU**$[MMA710]$**)**

April 17, 2009, morning (4 hours), v
Each problem is worth 3 points. No aids.
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If not otherwise stated W denotes a one-dimensional standard Brownian motion.

1. Let f be a real-valued continuous function on the unit interval $[0,1]$ and set

$$X=\int_0^1 f(t)dW(t)-\sum_{k=1}^{n} f(\frac{k-1}{n})(W(\frac{k}{n})-W(\frac{k-1}{n}))$$

where n is a fixed positive integer. Prove that

$$\mathrm{Var}(X)\leq \max_{|s-t|\leq\frac{1}{n}}(f(s)-f(t))^2.$$

Solution. Since $\int_0^1 f(t)dW(t)\in N(0,\int_0^1 f^2(t)dt)$, $\mathrm{Var}(X)=E\left[X^2\right]$. Moreover, introducing the function

$$g(t)=f(\frac{k-1}{n})\text{ if }\frac{k-1}{n}\leq t<\frac{k}{n},\ k=1,...,n$$

we have

$$\sum_{k=1}^{n} f(\frac{k-1}{n})(W(\frac{k}{n})-W(\frac{k-1}{n}))=\int_0^1 g(t)dW(t).$$

Hence

$$X=\int_0^1 (f(t)-g(t))dW(t)$$

and by the Itô isometry

$$E\left[X^2\right]=\int_0^1 (f(t)-g(t))^2dt=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(t)-f(\frac{k-1}{n}))^2dt$$

$$\leq \sum_{k=1}^{n} \frac{1}{n} \max_{\frac{k-1}{n} \leq t \leq \frac{k}{n}} (f(t) - f(\frac{k-1}{n}))^2 \leq \sum_{k=1}^{n} \frac{1}{n} \max_{|s-t| \leq \frac{1}{n}} (f(s) - f(t))^2$$
$$= \max_{|s-t| \leq \frac{1}{n}} (f(s) - f(t))^2.$$

2. Suppose T is a positive real number and

$$dX(t) = dt + \Delta(t) dW(t), \ 0 \leq t \leq T$$

where

$$\Delta(t) = \int_0^t \frac{1}{(t-s)^{\frac{1}{4}}} dW(s).$$

Find $E\left[X^2(T)\right]$ if $X(0) = 1$.

Solution. We have

$$X(T) = 1 + T + \int_0^T \Delta(t) dW(t).$$

Since

$$E\left[\Delta^2(t)\right] = \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds = 2t^{\frac{1}{2}}$$

it follows that

$$E\left[\int_0^T \Delta^2(t) dt\right] = \int_0^T E\left[\Delta^2(t)\right] dt = \frac{4}{3} T^{\frac{3}{2}}$$

and by the Itô isometry

$$E\left[\left\{\int_0^T \Delta(t) dW(t)\right\}^2\right] = E\left[\int_0^T \Delta^2(t) dt\right] = \frac{4}{3} T^{\frac{3}{2}}.$$

Now since

$$E\left[\int_0^T \Delta(t) dW(t)\right] = 0$$

it follows that

$$E\left[X^2(T)\right] = (1+T)^2 + \frac{4}{3} T^{\frac{3}{2}}.$$

3. Let $T > 0$ and consider a financial market with interest rate r and a stock price process $S = (S(t))_{0 \leq t \leq T}$ governed by a geometric Brownian motion with volatility $\sigma > 0$. A European derivative pays the amount $Y = | S(T) - K |$ at time of maturity T but the market fixes the prices of financial derivatives as in a Black-Scholes model with stock price volatility 1.02σ. Find an arbitrage.

Solution. First note that

$$(dS(t))^2 = \sigma^2 S^2(t)dt.$$

Let $u(t, S(t))$ be the market price of the derivative at time t. Since $| x | = 2x^+ - x$ for every real x, $u(t, S(t)) = 2c(t, S(t)) - S(t) + K$, where $c(t, S(t))$ is the European call price in a Black-Scholes model corresponding to volatility $\sigma_1 = 1.02\sigma$. Hence, if $\tau = T - t > 0$,

$$u'_x(t, S(t)) = 2N\left(\frac{\ln \frac{S(t)}{K} + (r + \frac{\sigma_1^2}{2})\tau}{\sigma_1 \sqrt{\tau}}\right) - 1$$

and

$$u''_{xx}(t, S(t)) = 2\varphi\left(\frac{\ln \frac{S(t)}{K} + (r + \frac{\sigma_1^2}{2})\tau}{\sigma_1 \sqrt{\tau}}\right)\frac{1}{\sigma_1 \sqrt{\tau} S(t)}$$

where $\varphi(y) = \exp(-\frac{y^2}{2})/\sqrt{2\pi}$. In paticular, $u''_{xx}(t, S(t)) > 0$ if $0 \leq t < T$.

Now short one derivative and long $u'_x(t, S(t))$ shares of stock at time t and consider the corresponding portfolio process $(X(t))_{0 \leq t \leq T}$ with $X(0) = 0$. Then

$$dX(t) = -u'_t(t, S(t))dt - u'_x(t, S(t))dS(t) - \frac{1}{2}u''_{xx}(t, S(t))(dS(t))^2$$

$$+u'_x(t, S(t))dS(t) + r\left\{X(t) + u(t, S(t)) - u'_x(t, S(t))S(t)\right\}dt$$

$$= \left\{-u'_t(t, S(t)) - \frac{\sigma^2 S^2(t)}{2}u''_{xx}(t, S(t)) + ru(t, S(t)) - rS(t)u'_x(t, S(t))\right\}dt + rX(t)dt$$

$$= \frac{(\sigma_1^2 - \sigma^2)S^2(t)}{2}u''_{xx}(t, S(t))dt + rX(t)dt.$$

From this

$$d(e^{-rt}X(t)) = e^{-rt}\frac{(\sigma_1^2-\sigma^2)S^2(t)}{2}u''_{xx}(t,S(t))dt$$

and

$$X(t) = e^{rt}\int_0^t e^{-r\lambda}\frac{(\sigma_1^2-\sigma^2)S^2(\lambda)}{2}u''_{xx}(\lambda,S(\lambda))d\lambda > 0$$

for $0 < t \leq T$ and we get an arbitrage.

Alternative solution. Let $c(t,S(t),K,T;\sigma)$ be the Black-Scholes price of a call with strike K and time of maturity T if the volatility of the stock price equals σ. Then

$$\text{vega} = \frac{\partial}{\partial\sigma}c(t,S(t),K,T;\sigma) = S(t)\varphi(\frac{\ln\frac{S(t)}{K}+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}})\sqrt{\tau} > 0$$

(see Options and Mathematics). Using the same notation as above

$$u(t,S(t)) = 2c(t,S(t),K,T;\sigma_1) - S(t) + K$$

and

$$\text{the Black-Scholes price } \Pi_Y(t) = 2c(t,S(t),K,T;\sigma) - S(t) + K.$$

Hence

$$u(0,S(0)) > \Pi_Y(0).$$

Since the derivative Y can be hedged in the time interval $[0,T]$ to the cost of the amount $\Pi_Y(0)$ at time zero, we sell the derivative Y on the market at time 0 and buy the hedging portfolio. Depositing the amount $u(0,S(0)) - \Pi_Y(0)$ in a bank at time 0 we get the profit $(u(0,S(0)) - \Pi_Y(0))\exp(rT) > 0$ at time T.

4. Prove that a market model does not admit arbitrage, if it has a risk-neutral probability measure.

5. Consider a market model possessing a unique risk-neutral measure $\tilde{P}$. Using standard notation $B(t,T) = \frac{1}{D(t)}\tilde{E}\left[D(T) \mid \mathcal{F}(t)\right]$, $\text{For}_S(t,T) = \frac{S(t)}{B(t,T)}$, and $\text{Fut}_S(t,T) = \tilde{E}\left[S(T) \mid \mathcal{F}(t)\right]$. Show that the forward-futures spread equals

$$\text{For}_S(0,T) - \text{Fut}_S(0,T) = \frac{\tilde{C}}{B(0,T)}$$

where $\tilde{C}$ is the covariance of $D(T)$ and $S(T)$ under the-risk neutral measure $\tilde{P}$. Conclude that $\text{For}_S(0,T)$ $=\text{Fut}_S(0,T)$ if the discount process is non-random and equal to $D(t) = e^{-rt}$ (an alternative proof of this property gives 1p).