SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

August 23, 2011, morning, v
No aids.
Questions on the exam: Oskar Hamlet, 0703 - 08 83 04
Each problem is worth 3 points.

1. Let $W(t)$, $t \geq 0$, be a one-dimensional Brownian motion. (a) Solve the stochastic differential equation

$$dX(t) = tX(t)dt + dW(t), \ t \geq 0$$

with the initial condition $X(0) = 1$. (b) Find $\mathrm{Cov}(X(s), X(t))$.

Solution. (a) We have

$$d(e^{-\frac{t^2}{2}} X(t)) = e^{-\frac{t^2}{2}} dW(t)$$

and as $X(0) = 1$,

$$e^{-\frac{t^2}{2}} X(t) - 1 = \int_0^t e^{-\frac{u^2}{2}} dW(u).$$

Solving for $X(t)$ we get

$$X(t) = e^{\frac{t^2}{2}} + e^{\frac{t^2}{2}} \int_0^t e^{-\frac{u^2}{2}} dW(u).$$

(b) Since

$$E\left[X(t)\right] = e^{\frac{t^2}{2}}$$

it follows that

$$\mathrm{Cov}(X(s), X(t)) = e^{\frac{s^2+t^2}{2}} E\left[\int_0^s e^{-\frac{u^2}{2}} dW(u) \int_0^t e^{-\frac{u^2}{2}} dW(u)\right].$$

Now if $s \le t$,

$$\mathrm{Cov}(X(s), X(t)) = e^{\frac{s^2+t^2}{2}} E\left[\int_0^t 1_{[0,s]} e^{-\frac{u^2}{2}} dW(u) \int_0^t e^{-\frac{u^2}{2}} dW(u)\right] =$$

$$e^{\frac{s^2+t^2}{2}} \int_0^t 1_{[0,s]} e^{-\frac{u^2}{2}} e^{-\frac{u^2}{2}} du = e^{\frac{s^2+t^2}{2}} \int_0^s e^{-u^2} du =$$

$$\sqrt{\pi} e^{\frac{s^2+t^2}{2}} \int_0^{\sqrt{2}s} e^{-\frac{v^2}{2}} \frac{dv}{\sqrt{2\pi}} = \sqrt{\pi} e^{\frac{s^2+t^2}{2}} (N(\sqrt{2}s) - \frac{1}{2}).$$

Thus for general $s, t \ge 0$,

$$\mathrm{Cov}(X(s), X(t)) = \sqrt{\pi} e^{\frac{s^2+t^2}{2}} (N(\sqrt{2}\min(s,t)) - \frac{1}{2}).$$

2. Let $W = (W_1(t), W_2(t))_{t \ge 0}$, be a two-dimensional Brownian motion, let $(\mathcal{F}(t))_{t \ge 0}$ be the filtration generated by W, and let

$$X = 3W_1(\frac{1}{3}) - W_1(\frac{1}{2}) - \frac{1}{2} W_2(\frac{1}{2}) + \frac{1}{2} W_2(1).$$

(a) Find a function $f(t)$, $0 \le t \le 1$, with its values in $\mathbf{R}^2$ such that

$$X = \int_0^1 f(t) \cdot dW(t).$$

(b) For any $t \ge 0$, find $E[X^2(t)]$, where $X(t) = E[X \mid \mathcal{F}(t)]$.

Solution. (a) Since $3W_1(\frac{1}{3}) - W_1(\frac{1}{2}) = 2W_1(\frac{1}{3}) + (-1)(W_1(\frac{1}{2}) - W_1(\frac{1}{3}))$ we get

$$3W_1(\frac{1}{3}) - W_1(\frac{1}{2}) = \int_0^1 f_1(t) dW_1(t)$$

where

$$f_1(t) = \begin{cases} 2, \ 0 \le t \le \frac{1}{3}, \\ -1, \ \frac{1}{3} < t \le \frac{1}{2}, \\ 0, \ \frac{1}{2} < t \le 1. \end{cases}$$

Moreover,

$$\frac{1}{2}W_2(1) - \frac{1}{2}W_2(\frac{1}{2}) = \int_0^1 f_2(t)dW_2(t)$$

where

$$f_2(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < t \le 1. \end{cases}$$

Thus $X = \int_0^1 f(t) \cdot dW(t)$ if

$$f(t) = (f_1(t), f_2(t)), \ 0 \le t \le 1.$$

(b) Since X is $\mathcal{F}_1$-measurable, $X(t) = X(1)$ if $t \ge 1$. Moreover,

$$X(t) = \int_0^t f(s) \cdot dW(s) \text{ if } 0 \le t \le 1.$$

Now if $t \ge 0$ we use the Itô isometry to get

$$E\left[X^2(t)\right] = E\left[\left(\int_0^{\min(1,t)} f_1(s)dW_1(s) + \int_0^{\min(1,t)} f_2(s)dW_2(s)\right)^2\right] =$$

$$E\left[\left(\int_0^{\min(1,t)} f_1(s)dW_1(s)\right)^2\right] + 2E\left[\int_0^{\min(1,t)} f_1(s)dW_1(s) \int_0^{\min(1,t)} f_2(s)dW_2(s)\right] +$$

$$E\left[\left(\int_0^{\min(1,t)} f_2(s)dW_2(s)\right)^2\right] = \int_0^{\min(1,t)} (f_1(s)^2 + f_2(s)^2)dt =$$

$$\begin{cases} 4t, \ 0 \le t \le \frac{1}{3}, \\ 1+t, \ \frac{1}{3} < t \le \frac{1}{2}, \\ \frac{11}{8} + \frac{t}{4}, \ \frac{1}{2} < t \le 1, \\ \frac{13}{8}, \ t > 1. \end{cases}$$

3. (Black-Scholes model) Two stock price processes S_1 and S_2 are governed by the stochastic differential equations

$$dS_i(t) = S_i(t)(\mu_i dt + \sigma_{i1}dW_1(t) + \sigma_{i2}dW_2(t)), \ t \ge 0, \ i = 1, 2$$

where $W = (W_1(t), W_2(t))_{t\geq 0}$ is a two-dimensional Brownian motion and where $\mu_i, \sigma_{i1}, \sigma_{i2} \in \mathbf{R}$, $i = 1,2$, and the volatility matrice $(\sigma_{ij})_{1\leq i,j\leq 2}$ is invertible. Moreover, the model has a bond with the the price $B(t) = B(0)e^{rt}$ at time t, where $B(0)$ and r are positive constants.

A financial derivative of European type pays the amount Y at time of maturity T, where

$$Y = (\frac{S_1^2(T)}{S_2(T)} - K)^+.$$

(a) Find the price $\Pi_Y(t) = v(t, S_1(t), S_2(t))$ of the derivative at time $t \in [0, T[$. (b) Compute the product

$$dS_1(t)\ dv(t, S_1(t), S_2(t)).$$

Solution. (a) Let $\sigma_i = (\sigma_{i1}, \sigma_{i2})$, $i = 1, 2$. We have

$$S_i(t) = S_i(0)e^{(\mu_i - \frac{|\sigma_i|^2}{2})t + \sigma_i\cdot W(t)},\ i = 1,2.$$

Hence, under the risk neutral measure $\tilde{P}$,

$$S_i(t) = S_i(0)e^{(r - \frac{|\sigma_i|^2}{2})t + \sigma_i\cdot \tilde{W}(t)},\ i = 1,2,$$

where $(\tilde{W}_1(t), \tilde{W}_2(t))_{0\leq t\leq T}$ is a two-dimensional Brownian motion. Moreover, the price of the derivative at time t is equal to

$$v(t, S_1(t), S_2(t)) = e^{-r\tau}\tilde{E}\left[(\frac{S_1^2(T)}{S_2(T)} - K)^+ \mid \mathcal{F}_t \right]$$

where $\mathcal{F}_t = \sigma(W(u);\ 0 \leq u \leq t)$. Thus, if $s_i = S_i(t)$, $i = 1,2$ and $\tau = T - t$,

$$v(t, s_1, s_2) = e^{-r\tau}\tilde{E}\left[(\frac{s_1^2}{s_2} e^{\left\{ 2(r - \frac{|\sigma_1|^2}{2}) - (r - \frac{|\sigma_2|^2}{2}) \right\}\tau + (2\sigma_1 - \sigma_2)\cdot(\tilde{W}(T) - \tilde{W}(t))} - K)^+ \right] =$$

$$e^{-r\tau}\tilde{E}\left[(\frac{s_1^2}{s_2} e^{\left\{ (r - |\sigma_1|^2 + \frac{|\sigma_2|^2}{2}) \right\}\tau + (2\sigma_1 - \sigma_2)\cdot(\tilde{W}(T) - \tilde{W}(t))} - K)^+ \right].$$

Under $\tilde{P}$ the random varible

$$(2\sigma_1 - \sigma_2)\cdot(\tilde{W}(T) - \tilde{W}(t))/\sqrt{\tau}$$

is centred Gaussian with standard deviation

$$\sigma =_{def} \sqrt{4 \mid \sigma_1 \mid^2 - 4\sigma_1 \cdot \sigma_2 + \mid \sigma_2 \mid^2}.$$

Now under $\tilde{P}$ suppose $G \in N(0,1)$. Then

$$v(t, s_1, s_2) = e^{-r\tau} \tilde{E} \left[(\frac{s_1^2}{s_2} e^{(\frac{\sigma^2}{2} - |\sigma_1|^2 + \frac{|\sigma_2|^2}{2})\tau} e^{\left\{(r - \frac{\sigma^2}{2}\right\}\tau + \sigma\sqrt{\tau}G} - K)^+ \right] =$$

$$e^{-r\tau} \tilde{E} \left[(\frac{s_1^2}{s_2} e^{|\sigma_1 - \sigma_2|^2\tau} e^{\left\{(r - \frac{\sigma^2}{2}\right\}\tau + \sigma\sqrt{\tau}G} - K)^+ \right]$$

and if we think of the Black-Scholes price of a standard call we get

$$v(t, s_1, s_2) =$$

$$\frac{s_1^2}{s_2} e^{|\sigma_1 - \sigma_2|^2\tau} \Phi(\frac{\ln \frac{s_1^2}{s_2 K} e^{|\sigma_1 - \sigma_2|^2\tau} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - K e^{-r\tau} \Phi(\frac{\ln \frac{s_1^2}{s_2 K} e^{|\sigma_1 - \sigma_2|^2\tau} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) =$$

$$\frac{s_1^2}{s_2} e^{|\sigma_1 - \sigma_2|^2\tau} \Phi(\frac{\ln \frac{s_1^2}{s_2 K} + (r + \mid \sigma_1 - \sigma_2 \mid^2 + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - K e^{-r\tau} \Phi(\frac{\ln \frac{s_1^2}{s_2 K} + (r + \mid \sigma_1 - \sigma_2 \mid^2 - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

(b) For short we write $v_i'(t) = v_{s_i}'(t, S_1(t), S_2(t))$ and $v_{ij}''(t) = v_{s_i s_j}''(t, S_1(t), S_2(t))$ for $i, j = 1, 2$. Then

$$dv(t, S_1(t), S_2(t)) = v_t'(t, S_1(t), S_2(t))dt + \sum_{i=1}^{2} v_i'(t) S_i(t)(\mu_i dt + \sigma_{i1} dW_1(t) + \sigma_{i2} dW_2(t)) +$$

$$\frac{1}{2} \sum_{i,j=1}^{2} v_{ij}''(t) S_i(t) S_j(t)(\mu_i dt + \sigma_{i1} dW_1(t) + \sigma_{i2} dW_2(t))(\mu_j dt + \sigma_{j1} dW_1(t) + \sigma_{j2} dW_2(t)) =$$

$$(...)dt + \sum_{i=1}^{2} v_i'(t) S_i(t)(\sigma_{i1} dW_1(t) + \sigma_{i2} dW_2(t)).$$

Hence

$$dS_1(t)\ dv(t, S_1(t), S_2(t)) =$$

$$S_1(t)(\mu_1 dt + \sigma_{11} dW_1(t) + \sigma_{12} dW_2(t)) \left((...)dt + \sum_{i=1}^{2} S_i(t) v_i'(t)(\sigma_{i1} dW_1(t) + \sigma_{i2} dW_2(t)) \right) =$$

$$S_1(t)\left\{\sum_{i=1}^{2} S_i(t)v_i'(t)(\sigma_{11}\sigma_{i1}+\sigma_{12}\sigma_{i2})\right\} dt.$$

It remains to compute $v_1'(t)$ and $v_2'(t)$.

Here if N is the standard normal distribution function a computation yields

$$\frac{\partial}{\partial s}\left\{sN(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})-Ke^{-r\tau}N(\frac{\ln\frac{s}{K}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right\}=N(\frac{\ln\frac{s}{K}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

and accordingly from this

$$v_1'(t)=v_{s_1}'(t,s_1,s_2)=\frac{2s_1}{s_2}e^{|\sigma_1-\sigma_2|^2\tau}N(\frac{\ln\frac{s_1^2}{s_2K}e^{|\sigma_1-\sigma_2|^2\tau}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})=$$

$$\frac{2s_1}{s_2}e^{|\sigma_1-\sigma_2|^2\tau}N(\frac{\ln\frac{s_1^2}{s_2K}+(r+\mid\sigma_1-\sigma_2\mid^2+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})$$

and

$$\begin{aligned} v_2'(t) &= v_{s_2}'(t,s_1,s_2)=-\frac{s_1^2}{s_2^2}e^{|\sigma_1-\sigma_2|^2\tau}N(\frac{\ln\frac{s_1^2}{s_2K}e^{|\sigma_1-\sigma_2|^2\tau}+(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})= \\ & -\frac{s_1^2}{s_2^2}e^{|\sigma_1-\sigma_2|^2\tau}N(\frac{\ln\frac{s_1^2}{s_2K}+(r+\mid\sigma_1-\sigma_2\mid^2+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}). \end{aligned}$$

4. Let $W=(W(t))_{t\geq 0}$ be a one-dimensional Brownian motion with filtration $(\mathcal{F}(t))_{t\geq 0}$. (a) Prove that W is a martingale. (b) Suppose $\sigma\in\mathbf{R}$. Prove that the process $Z(t)=\exp\left\{\sigma W(t)-\frac{\sigma^2}{2}t\right\}$, $t\geq 0$, is a martingale.

5. Suppose $W=(W(t))_{t\geq 0}$ is a one-dimensional Brownian motion and consider the stochastic differential equation

$$dX(u)=\beta(u,X(u))du+\gamma(u,X(u))dW(u)$$

where $\beta(t,x)$ and $\gamma(t,x)$ are real-valued functions.

Let T be a positive constant and $h(x)$ a Borel measurable real-valued function such that $E^{t,x}\left[\mid h(X(T)) \mid\right] < \infty$ for all $t \in [0, T]$ and x. Prove that the function

$$f(t,x) = E^{t,x}\left[e^{-r(T-t)}h(X(T))\right]$$

solves the partial differential equation

$$u'_t(t,x) + \beta(t,x)u'_x(t,x) + \frac{1}{2}\gamma^2(t,x)u''_{xx}(t,x) = ru(t,x).$$