SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

December 13, 2011, morning, V
No aids.
Questions on the exam: Christer Borell, 0705 292322
Each problem is worth 3 points.

1. Let $W = (W_1(t), W_2(t))_{0\leq t\leq T}$ be a two-dimensional standard Brownian motion and $(\mathcal{F}(t))_{0\leq t\leq T}$ a filtration for W. Find an adapted process $(\Gamma(t))_{0\leq t\leq T}$ such that

$$W_1^2(t) - W_2^2(t) = \int_0^t \Gamma(u)\cdot dW(u),\ 0\leq t\leq T.$$

Solution. We have

$$d(W_1^2(t) - W_2^2(t)) = dW_1^2(t) - dW_2^2(t) = 2W_1(t)dW_1(t) + \frac{1}{2}2(dW_1(t))^2$$

$$-2W_2(t)dW_2(t) - \frac{1}{2}2(dW_2(t))^2 = 2W_1(t)dW_1(t) + dt - 2W_2(t)dW_2(t) - dt$$
$$= 2W_1(t)dW_1(t) - 2W_2(t)dW_2(t)$$

and

$$W_1^2(t) - W_2^2(t) = \int_0^t (2W_1(u), -2W_2(u))\cdot dW(u).$$

Thus $\Gamma(t) = (2W_1(t), -2W_2(t)), 0\leq t\leq T$, is a solution of the the problem.

2. (Black-Scholes model for two stocks with non-singular volatility matrice $\sigma = (\sigma_{ij})_{1\leq i,j\leq 2}$ and interest rate $r > 0$) Suppose $K, T > 0$ and consider a financial derivative of European type with the payoff

$$Y = (\frac{S_1(T)}{S_2(T)} - K)^+$$

at time of maturity T. (a) Find the price $\Pi_Y(t)$ of the derivative at time $t \in [0, T[$. (b) Find adapted processes $(\Delta_i(t))_{0 \le t < T}$, $i = 1, 2$, such that

$$d\Pi_Y(t) = \sum_{i=1}^{2} \Delta_i(t) dS_i(t) + r \left\{ \Pi_Y(t) - \sum_{i=1}^{2} \Delta_i(t) S_i(t) \right\} dt$$

for $0 \le t < T$.

Solution. (a) With $\sigma_i = (\sigma_{i1,}\sigma_{i2})$ and standard notation, we have

$$S_i(t) = S_i(0) e^{(r - \frac{|\sigma_i|^2}{2})t + \sigma_i \cdot \tilde{W}(t)}, \; i = 1, 2.$$

Hence, if $\tau = T - t > 0$,

$$\Pi_Y(t) = e^{-r\tau} \tilde{E} \left[(\frac{S_1(T)}{S_2(T)} - K)^+ \mid \mathcal{F}(t) \right]$$

$$= e^{-r\tau} \tilde{E} \left[(\frac{S_1(t)}{S_2(t)} e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + (\sigma_1 - \sigma_2) \cdot (\tilde{W}(T) - \tilde{W}(t))} - K)^+ \mid \mathcal{F}(t) \right]$$

$$= e^{-r\tau} \tilde{E} \left[(s e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + \sqrt{\tau}(\sigma_1 - \sigma_2) \cdot \tilde{W}(1)} - K)^+ \right]_{s = \frac{S_1(t)}{S_2(t)}}$$

$$= e^{-r\tau} E \left[(s e^{(\frac{|\sigma_2|^2}{2} - \frac{|\sigma_1|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2| G} - K)^+ \right]_{s = \frac{S_1(t)}{S_2(t)}}$$

where $G \in N(0, 1)$. Thus, if

$$\hat{r} - \frac{| \sigma_1 - \sigma_2 |^2}{2} = \frac{| \sigma_2 |^2}{2} - \frac{| \sigma_1 |^2}{2}$$

we have

$$\hat{r} = | \sigma_2 |^2 - \sigma_1 \cdot \sigma_2$$

and

$$\Pi_Y(t) = a e^{-\hat{r}\tau} E \left[(s e^{(\hat{r} - \frac{|\sigma_1 - \sigma_2|^2}{2})\tau + \sqrt{\tau}|\sigma_1 - \sigma_2| G} - K)^+ \right]_{s = \frac{S_1(t)}{S_2(t)}}$$

where

$$a = e^{(\hat{r} - r)\tau}.$$

Now by the Black-Scholes call price formula

$$\Pi_Y(t) = v(t, S_1(t), S_2(t)) = a\left(\frac{S_1(t)}{S_2(t)}\Phi(d_+(\tau, \frac{S_1(t)}{S_2(t)})) - Ke^{-\hat{r}\tau}\Phi(d_-(\tau, \frac{S_1(t)}{S_2(t)}))\right)$$

where

$$d_\pm(\tau, \frac{S_1(t)}{S_2(t)}) = \frac{\ln\frac{S_1(t)}{KS_2(t)} + (\hat{r} \pm \frac{|\sigma_1-\sigma_2|^2}{2})\tau}{\mid \sigma_2 - \sigma_2 \mid \sqrt{\tau}}$$

that is

$$d_+(\tau, \frac{S_1(t)}{S_2(t)}) = \frac{\ln\frac{S_1(t)}{KS_2(t)} + (\frac{|\sigma_1|^2}{2} + \frac{3|\sigma_2|^2}{2} - 2\sigma_1 \cdot \sigma_2)\tau}{\mid \sigma_2 - \sigma_1 \mid \sqrt{\tau}}$$

and

$$d_-(\tau, \frac{S_1(t)}{S_2(t)}) = \frac{\ln\frac{S_1(t)}{KS_2(t)} + (-\frac{|\sigma_1|^2}{2} + \frac{|\sigma_2|^2}{2})\tau}{\mid \sigma_2 - \sigma_1 \mid \sqrt{\tau}}.$$

(b) By martingale representation the derivative may be replicated and

$$d\Pi_Y(t) = v'_{s_1}(t, S_1(t), S_2(t))dS_1(t) + v'_{s_2}(t, s_1, S_2(t))dS_2(t) + (...)dt.$$

Thus remembering the delta of a standard call,

$$\Delta_1(t) = v'_{s_1}(t, S_1(t), S_2(t)) = \frac{a}{S_2(t)}\Phi(d_+(\tau, \frac{S_1(t)}{S_2(t)}))$$

and

$$\Delta_2(t) = v'_{s_1}(t, S_1(t), S_2(t)) = -\frac{aS_1(t)}{S_2(t)^2}\Phi(d_+(\tau, \frac{S_1(t)}{S_2(t)})).$$

3. An interest rate process $(R(t))_{t\geq 0}$ is governed by the equation

$$dR(t) = a(b - R(t))dt + \sigma dW(t),\ t \geq 0$$

where $W(t)$, $t \geq 0$, is a one-dimensional Brownian motion and a, b, and σ positive constants.

Suppose $R(0) < b$ and let $\tau = \min\{t > 0;\ R(t) = b\}$. Find $P[\tau \leq T]$ if $T > 0$.

(Hint: Show that there is a strictly increasing function $f(v)$, $v \geq 0$, such that the process

$$B(v) = \int_0^{f(v)} e^{at} dW(t), \ v \geq 0$$

is a standard Brownian motion.)

Solution. Set $X(t) = R(t) - b$ so that

$$dX(t) = -aX(t)dt + \sigma dW(t), \ t \geq 0$$

and

$$X(t) = e^{-at}(X(0) + \sigma \int_0^t e^{au} dW(u)).$$

Thus

$$\tau = \min \left\{ t > 0; \ \int_0^t e^{au} dW(u) = \frac{1}{\sigma}(b - R(0)) \right\}.$$

But

$$Y(t) =_{def} \int_0^t e^{au} dW(u) \in N(0, f^{-1}(t))$$

with $v = f^{-1}(t) = \frac{1}{2a}(e^{2at} - 1)$, $0 \leq t < \infty$, and $(Y(t))_{t \geq 0}$ is a centred Gaussian process with covariance

$$E\left[Y(s)Y(t)\right] = E\left[\int_0^s e^{au} dW(u) \int_0^t e^{au} dW(u)\right]$$

$$= E\left[\int_0^{\min(s,t)} e^{au} dW(u) \int_0^{\min(s,t)} e^{au} dW(u)\right] = \mathrm{Var}(Y(\min(s,t)))$$

$$= f^{-1}(\min(s,t)) = \min(f^{-1}(s), f^{-1}(t))$$

where the third equality follows from the Itô isometry. Now

$$B(v) = Y(f(v)) = Y(\frac{1}{2a} \ln(1 + 2av)), \ v \geq 0$$

is a centred Gaussian process with covariance

$$E\left[B(u)B(v)\right] = \min(u, v)$$

and we conclude that the process B is a standard Brownian motion. Moreover, if

$$c = \frac{1}{\sigma}(b - R(0)),$$

the Bachelier double law yields

$$P\left[\tau \le T\right] = P\left[\max_{0 \le t \le T} Y(t) \ge c\right]$$

$$= P\left[\max_{0 \le v \le \frac{1}{2a}(e^{2aT}-1)} B(v) \ge c\right] = 2(1 - \Phi(\frac{c}{\sqrt{\frac{1}{2a}(e^{2aT}-1)}}))$$

$$= 2\Phi(\frac{R(0) - b}{\sigma\sqrt{\frac{1}{2a}(e^{2aT}-1)}}).$$

4. Let W be a one-dimensional standard Brownian motion and set

$$Q = \sum_{i=1}^{n}(W(t_i) - W(t_{i-1}))^2$$

where $0 = t_0 < t_1 < ...t_{n-1} < t_n = T < \infty$. Show that $E\left[Q\right] = T$ and

$$\mathrm{Var}(Q) \le 2T \max_{1 \le i \le n}(t_i - t_{i-1}).$$

5. Let W be a one-dimensional standard Brownian motion and suppose $m > 0$ and $\tau_m = \min\left\{t > 0;\ W(t) = m\right\}$. Use the formula

$$P\left[\tau_m \le t,\ W(t) \le w\right] = P\left[W(t) \ge 2m - w\right],\ w \le m,$$

to prove that τ_m has the density

$$f_m(t) = \frac{m}{t\sqrt{2\pi t}}e^{-\frac{m^2}{2t}},\ t > 0.$$