SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

April 13, 2012, morning, v
No aids.
Questions on the exam: Christer Borell, 0705 292322
Each problem is worth 3 points.

1. A stock price process $(S(t))_{0\leq t\leq T}$ is governed by the equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)),\ 0 \leq t \leq T,$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$ are constants and $(W(t))_{0\leq t\leq T}$ is a one-dimensional standard Brownian motion. Find

$$E\left[W(T)\int_0^T S(t)dW(t)\right].$$

Solution. The Itô isometry gives

$$E\left[W(T)\int_0^T S(t)dW(t)\right] = E\left[\int_0^T 1dW(t)\int_0^T S(t)dW(t)\right]$$

$$= \int_0^T E\left[1\cdot S(t)\right]dt = \int_0^T E\left[S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W(t)}\right]dt$$

$$= S(0)\int_0^T e^{\mu t}dt = \begin{cases} \frac{S(0)}{\mu}(e^{\mu T}-1) \text{ if } \mu \neq 0, \\ S(0)T \text{ if } \mu = 0. \end{cases}$$

2. Suppose $W(t) = (W_1(t), W_2(t))$, $t \geq 0$ is a 2-dimensional standard Brownian motion. Compute (a) $E\left[W_1(1) \mid W_1(1)+W_2(1)\right]$, (b) $E\left[W_1(1)^2 \mid W_1(1)+W_2(1)\right]$ (c) $E\left[W_1(1)W_2(1) \mid W_1(1)+W_2(1)\right]$.

Solution. To simplify notation set $X = W_1(1)$ and $Y = W_2(1)$ and note that $X, Y \in N(0,1)$ are independent.

(a) The random variables $X+Y$ and $X-Y$ are independent since the random vector $(X+Y, X-Y)$ is Gaussian and

$$\mathrm{Cov}(X+Y, X-Y) = E\left[X^2 - Y^2\right] = 0.$$

Moreover,

$$X = \frac{1}{2}(X+Y) + \frac{1}{2}(X-Y)$$

and we get

$$E\left[W_1(1) \mid W_1(1) + W_2(1)\right]$$

$$= E\left[X \mid X+Y\right] = \frac{1}{2}E\left[(X+Y) + (X-Y) \mid X+Y\right]$$

$$= \frac{1}{2}E\left[X+Y \mid X+Y\right] + \frac{1}{2}E\left[X-Y \mid X+Y\right]$$

$$\frac{1}{2}(X+Y) + \frac{1}{2}E\left[X-Y\right] = \frac{1}{2}(X+Y) = \frac{1}{2}(W_1(1) + W_2(1)).$$

(b) We have

$$E\left[W_1(1)^2 \mid W_1(1) + W_2(1)\right] = E\left[X^2 \mid X+Y\right]$$

$$= \frac{1}{4}E\left[(X+Y)^2 \mid X+Y\right] + \frac{1}{2}E\left[(X+Y)(X-Y) \mid X+Y\right] + \frac{1}{4}E\left[(X-Y)^2 \mid X+Y\right]$$

$$= \frac{1}{4}(X+Y)^2 + \frac{1}{2}(X+Y)E\left[X-Y \mid X+Y\right] + \frac{1}{4}E\left[(X-Y)^2\right]$$

$$= \frac{1}{4}(X+Y)^2 + \frac{1}{2}(X+Y)E\left[X-Y\right] + \frac{1}{2}$$

$$= \frac{1}{4}(X+Y)^2 + \frac{1}{2} = \frac{1}{4}(W_1(1) + W_2(1))^2 + \frac{1}{2}.$$

(c) We have

$$E\left[W_1(1)W_2(1) \mid W_1(1) + W_2(1)\right]$$

$$= E\left[XY \mid X+Y\right] = \frac{1}{4}E\left[(X+Y)^2 \mid X+Y\right] - \frac{1}{4}E\left[(X-Y)^2 \mid X+Y\right]$$

$$= \frac{1}{4}(X+Y)^2 - \frac{1}{4}E\left[(X-Y)^2\right] = \frac{1}{4}(X+Y)^2 - \frac{1}{2}$$

$$= \frac{1}{4}(W_1(1) + W_2(1))^2 - \frac{1}{2}.$$

3. Let $S_f(t)$ denote the price in foreign currency of a foreign stock and $Q(t)$ the exchange rate, which gives units of domestic currency per unit of foreign currency. Moreover, suppose K, L, and T are strictly positive real numbers and consider a European derivative which pays the amount

$$Y = \begin{cases} L \text{ if } S_f(T) \geq K, \\ 0 \text{ if } S_f(T) < K, \end{cases}$$

in domestic currency at time of maturity T. Derive the price $\Pi_Y(0)$ of the derivative in domestic currency at time 0 under the following assumptions:

(i) The domestic and foreign interest rates are constant and denoted by r and r_f, respectively.

(ii) There exists a 2-dimensional standard Brownian motion $W = (W_1, W_2)$ such that

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f} dt + \sigma_{11} dW_1(t) + \sigma_{12} dW_2(t)), \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t)), \end{cases}$$

where $\alpha_{S_f}, \alpha_Q, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \in \mathbf{R}$ and the volatility matrice $(\sigma_{ik})_{1 \leq i,k \leq 2}$ is invertible.

Solution. If $\sigma_i = [\sigma_{i1} \ \sigma_{i2}]$, $i = 1, 2$,

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f} dt + \sigma_1 \cdot dW(t)), \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_2 \cdot dW(t)). \end{cases}$$

Let $B_f(t) = B_f(0)e^{r_f t}$ be the foreign bond price at time t and

$$\begin{cases} S(t) = Q(t)S_f(t) \text{ if } 0 \leq t \leq T, \\ U(t) = Q(t)B_f(t) \text{ if } 0 \leq t \leq T. \end{cases}$$

Now $(S(t))_{0 \leq t \leq T}$ and $(U(t))_{0 \leq t \leq T}$ can be viewed as price processes of domestic assets and

$$\begin{cases} dS(t) = S(t)((\cdot)dt + (\sigma_1 + \sigma_2) \cdot dW(t)), \\ dU(t) = U(t)((\cdot)dt + \sigma_2 \cdot dW(t)), \end{cases}$$

for appropriate drift coefficients which need not be specified. If $\tilde{P}$ denotes the domestic risk-neutral measure we have

$$\begin{cases} dS(t) = S(t)(rdt + (\sigma_1 + \sigma_2) \cdot d\tilde{W}(t)), \\ dU(t) = U(t)(rdt + \sigma_2 \cdot d\tilde{W}(t)), \end{cases}$$

where under $\tilde{P}$ the process $\tilde{W}$ is a 2-dimensional standard Brownian motion. Moreover,

$$Y = L1_{\left[S_f(T)\geq K\right]} = L1_{\left[\frac{S(T)}{U(T)}\geq \frac{K}{B_f(T)}\right]}.$$

Here

$$\frac{S(t)}{U(t)} = \frac{S_f(0)}{B_f(0)}e^{\frac{1}{2}(|\sigma_2|^2-|\sigma_1+\sigma_2|^2)t+\sigma_1\cdot\tilde{W}(t)}.$$

Now

$$\Pi_Y(0) = Le^{-rT}\tilde{E}\left[1_{\left[\frac{S(T)}{U(T)}\geq\frac{K}{B_f(T)}\right]}\right]$$

$$= Le^{-rT}\tilde{P}\left[\frac{S_f(0)}{B_f(0)}e^{\frac{1}{2}(|\sigma_2|^2-|\sigma_1+\sigma_2|^2)T+\sigma_1\cdot\tilde{W}(T)} \geq \frac{K}{B_f(T)}\right]$$

$$= Le^{-rT}\tilde{P}\left[e^{\sigma_1\cdot\tilde{W}(T)} \geq \frac{KB_f(0)}{B_f(T)S_f(0)}e^{-\frac{1}{2}(|\sigma_2|^2-|\sigma_1+\sigma_2|^2)T}\right]$$

$$= Le^{-rT}\tilde{P}\left[\sigma_1\cdot\tilde{W}(T)\geq \ln(\frac{K}{S_f(0)}) - (r_f+\frac{1}{2}(\mid\sigma_2\mid^2-\mid\sigma_1+\sigma_2\mid^2)T\right]$$

$$= Le^{-rT}\Phi\left(\frac{1}{\mid\sigma_1\mid\sqrt{T}}\left\{(r_f+\frac{1}{2}(\mid\sigma_2\mid^2-\mid\sigma_1+\sigma_2\mid^2)T-\ln(\frac{K}{S_f(0)})\right\}\right)$$

$$= Le^{-rT}\Phi\left(\frac{1}{\mid\sigma_1\mid\sqrt{T}}\left\{(r_f-(\sigma_1\cdot\sigma_2+\frac{1}{2}\mid\sigma_1\mid^2)T-\ln(\frac{K}{S_f(0)})\right\}\right).$$

4. Let $W=(W(t))_{t\geq 0}$ be a one-dimensional standard Brownian motion with a filtration $(\mathcal{F}(t))_{t\geq 0}$. (a) Prove that W is a martingale. (b) Suppose $\sigma\in\mathbf{R}$. Prove that the process $Z(t)=\exp\left\{\sigma W(t)-\frac{\sigma^2}{2}t\right\}$, $t\geq 0$, is a martingale.

5. Consider a market model possessing a unique risk-neutral measure $\tilde{P}$. Using standard notation, $B(t,T)=\frac{1}{D(t)}\tilde{E}\left[D(T)\mid\mathcal{F}(t)\right]$, $\mathrm{For}_S(t,T)=\frac{S(t)}{B(t,T)}$, and $\mathrm{Fut}_S(t,T)=\tilde{E}\left[S(T)\mid\mathcal{F}(t)\right]$. (a) Show that the forward-futures spread equals

$$\mathrm{For}_S(0,T)-\mathrm{Fut}_S(0,T)=\frac{\tilde{C}}{B(0,T)},$$

where $\tilde{C}$ is the covariance of $D(T)$ and $S(T)$ under the risk-neutral measure $\tilde{P}$. (b) Prove that $\mathrm{For}_S(0,T)=\mathrm{Fut}_S(0,T)$ if the discount process $(D(t))_{0\leq t\leq T}$ is nonrandom.