SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

August 28, 2012, morning, v
No aids.
Questions on the exam: Christer Borell, 0705 292322
Each problem is worth 3 points.

1. Let $W = (W_1, ..., W_n)$ be an n-dimensional standard Brownian motion and put

$$X = W_1(T) - \sum_{k=1}^{n} W_k^2(T),$$

where $T > 0$ is fixed. Compute $E\left[e^{aX}\right]$ for every $a \in \mathbf{R}$.

Solution. We have

$$E\left[e^{aX}\right] = E\left[e^{a(W_1(T)-W_1^2(T))}\right] E\left[e^{-a\Sigma_{k=2}^{n} W_k^2(T)}\right]$$

$$= E\left[e^{a(W_1(T)-W_1^2(T))}\right] \left(E\left[e^{-aW_1^2(T)}\right]\right)^{n-1}.$$

Here, if $a > -\frac{1}{2T}$,

$$E\left[e^{a(W_1(T)-W_1^2(T))}\right] = \int_{-\infty}^{\infty} e^{a(\sqrt{T}x - Tx^2) - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{a\sqrt{T}x - \frac{2aT+1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{2aT+1}} \int_{-\infty}^{\infty} e^{\frac{a\sqrt{T}}{\sqrt{2aT+1}}y - \frac{1}{2}y^2} \frac{dy}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2aT+1}} e^{\frac{1}{2}\frac{a^2T}{2aT+1}}$$

and if $a \leq -\frac{1}{2T}$ we see from the above that

$$E\left[e^{a(W_1(T)-W_1^2(T))}\right] = \infty.$$

Moreover, if $a > -\frac{1}{2T}$,

$$E\left[e^{-aW_1^2(T)}\right] = \int_{-\infty}^{\infty} e^{-aTx^2 - \frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{2aT+1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{2aT+1}}$$

and if $a \leq -\frac{1}{2T}$,

$$E\left[e^{-aW_1^2(T)}\right] = \infty.$$

Thus

$$E\left[e^{aX}\right] = \begin{cases} \frac{1}{(2aT+1)^{\frac{n}{2}}} e^{\frac{1}{2}\frac{a^2T}{2aT+1}} \text{ if } a > -\frac{1}{2T}, \\ \infty \text{ if } a \leq -\frac{1}{2T}. \end{cases}$$

2. (Vasicek interest rate model) Suppose $R(0) = r$ and

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t), \; t \geq 0,$$

where $\alpha, \beta, \sigma > 0$ and $r \in \mathbf{R}$ are parameters and W is a standard 1-dimensional Brownian motion.

(a) Find the covariance of $W(t)$ and $R(t)$.
(b) Find the covariance of $W^2(t)$ and $R(t)$.

Solution. (a) We have

$$d(e^{\beta t}R(t)) = \alpha e^{\beta t}dt + \sigma e^{\beta t}dW(t)$$

and

$$R(t) = re^{-\beta t} + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \int_0^t \sigma e^{-\beta(t-u)}dW(u).$$

Hence

$$R(t) - E\left[R(t)\right] = \int_0^t \sigma e^{-\beta(t-u)}dW(u)$$

and

$$\mathrm{Cov}(W(t), R(t)) = E\left[\int_0^t 1dW(u) \int_0^t \sigma e^{-\beta(t-u)}dW(u)\right]$$

$$= \int_0^t 1 \cdot \sigma e^{-\beta(t-u)} du = \frac{\sigma}{\beta}(1 - e^{-\beta t}).$$

(b) Since

$$\int_0^t W(u)dW(u) = \frac{1}{2}W^2(t) - \frac{t}{2}$$

it follows that

$$W^2(t) = 2\int_0^t W(u)dW(u) + t.$$

Thus

$$W^2(t) - E\left[W^2(t)\right] = \int_0^t 2W(u)dW(u)$$

and

$$\mathrm{Cov}(W^2(t), R(t)) = E\left[\int_0^t 2W(u)dW(u) \int_0^t \sigma e^{-\beta(t-u)} dW(u)\right]$$

$$= \int_0^t E\left[2W(u)\sigma e^{-\beta(t-u)}\right] du = \int_0^t 0dt = 0.$$

3. (Black-Scholes model) Suppose $T > 0$. A financial derivative of European type pays the amount

$$Y =\mid \frac{S(\frac{T}{2})}{S(0)} - \frac{S(T)}{S(\frac{T}{2})} \mid$$

at time of maturity T. Find $\Pi_Y(0)$.

Proof. First recall that $\mid x - y \mid = 2\max(x, y) - x - y$. Hence

$$\Pi_Y(0) = e^{-rT}\tilde{E}\left[Y\right]$$

$$= 2e^{-rT}\tilde{E}\left[\max(\frac{S(\frac{T}{2})}{S(0)}, \frac{S(T)}{S(\frac{T}{2})})\right] - e^{-rT}\tilde{E}\left[\frac{S(\frac{T}{2})}{S(0)}\right] - e^{-rT}\tilde{E}\left[\frac{S(T)}{S(\frac{T}{2})}\right]$$

$$= 2e^{-rT}\tilde{E}\left[\max(\frac{S(\frac{T}{2})}{S(0)}, \frac{S(T)}{S(\frac{T}{2})})\right] - 2e^{-\frac{rT}{2}}.$$

From now on to simplify notation put $S(0) = s$, $a = \frac{T}{2}$, and

$$Z = \max(\frac{S(\frac{T}{2})}{S(0)}, \frac{S(T)}{S(\frac{T}{2})}).$$

If W denotes a standard Brownian motion under P we have

$$e^{-rT}\tilde{E}\left[Z\right] = e^{-rT}E\left[\max(\frac{se^{(r-\frac{\sigma^2}{2})a+\sigma W(a)}}{s}, \frac{se^{(r-\frac{\sigma^2}{2})T+\sigma W(T)}}{se^{(r-\frac{\sigma^2}{2})a+\sigma W(a)}})\right]$$

$$= e^{-rT}E\left[\max(e^{(r-\frac{\sigma^2}{2})a+\sigma W(a)}, e^{(r-\frac{\sigma^2}{2})a+\sigma(W(T)-W(a))})\right]$$

$$= e^{-(r+\frac{\sigma^2}{2})a}E\left[\max(e^{\sigma\sqrt{a}G}, e^{\sigma\sqrt{a}H})\right]$$

$$= e^{-(r+\frac{\sigma^2}{2})a}E\left[e^{\sigma\sqrt{a}\max(G,H)}\right],$$

where under P the random variables $G, H \in N(0,1)$ are independent. Moreover,

$$P\left[\max(G,H) \leq x\right] = P\left[G \leq x,\ H \leq x\right]$$

$$= P\left[G \leq x\right]P\left[\ H \leq x\right] = \Phi^2(x).$$

Hence

$$E\left[e^{\sigma\sqrt{a}\max(G,H)}\right] = \int_{-\infty}^{\infty} e^{\sigma\sqrt{a}x}\frac{d}{dx}\Phi^2(x)dx$$

$$= 2\int_{-\infty}^{\infty} e^{\sigma\sqrt{a}x}\Phi(x)\varphi(x)dx.$$

Now introduce $b = \sigma\sqrt{a}$ and note that

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}}\int_{-\infty}^{\infty}\Phi(x)\varphi(x-b)dx$$

$$= e^{\frac{b^2}{2}}\int_{-\infty}^{\infty}\Phi(b-x)\varphi(x)dx.$$

But

$$\int_{-\infty}^{\infty}\varphi(y-x)\varphi(x)dx = \frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})$$

since $G + H \in N(0, 2)$ and by integration from $y = -\infty$ to $y = b$ we get

$$\int_{-\infty}^{\infty} \Phi(b - x)\varphi(x)dx = \int_{-\infty}^{b} \frac{1}{\sqrt{2}}\varphi(\frac{y}{\sqrt{2}})dy = \Phi(\frac{b}{\sqrt{2}}).$$

Hence

$$\int_{-\infty}^{\infty} e^{bx}\Phi(x)\varphi(x)dx = e^{\frac{b^2}{2}}\Phi(\frac{b}{\sqrt{2}})$$

and

$$e^{-rT}\tilde{E}\left[Z\right] = e^{-(r+\frac{\sigma^2}{2})a}E\left[e^{\sigma\sqrt{a}\max(G,H)}\right] = 2e^{-(r+\frac{\sigma^2}{2})a}\int_{-\infty}^{\infty} e^{\sigma\sqrt{a}x}\Phi(x)\varphi(x)dx$$

$$= 2e^{-(r+\frac{\sigma^2}{2})a}e^{\frac{\sigma^2 a}{2}}\Phi(\frac{\sigma\sqrt{a}}{\sqrt{2}}) = 2e^{-\frac{rT}{2}}\Phi(\frac{\sigma\sqrt{T}}{2}).$$

Thus

$$\Pi_Y(0) = 4e^{-r\frac{T}{2}}\Phi(\frac{\sigma\sqrt{T}}{2}) - 2e^{-\frac{rT}{2}}$$

$$= e^{-\frac{rT}{2}}(4\Phi(\frac{\sigma\sqrt{T}}{2}) - 2).$$

4. (Cox-Ingersoll-Ross interest rate model) Suppose $R(0) = r$ and

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t),\ t \geq 0,$$

where $\alpha, \beta, \sigma > 0$ and $r \in \mathbf{R}$ are parameters and W is a standard 1-dimensional Brownian motion. Prove that

$$E\left[R(t)\right] = e^{-\beta t}r + \frac{\alpha}{\beta}(1 - e^{-\beta t}).$$

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and Z a random variable such that $P\left[Z > 0\right] = 1$ and $E\left[Z\right] = 1$. Set

$$\tilde{P}(A) = \int_A Z(\omega)dP(\omega).$$

Furthermore, suppose $(\mathcal{F}(t))_{0\leq t\leq T}$ is a filtration and $Z(t) = E\left[Z \mid \mathcal{F}(t)\right]$, $0 \leq t \leq T$.

Now let $0 \leq s \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Show that

$$\tilde{E}\left[Y \mid \mathcal{F}(s)\right] = \frac{1}{Z(s)}E\left[YZ(t) \mid \mathcal{F}(s)\right].$$