SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

December 18, 2012, morning, H
No aids.
Questions on the exam: Christer Borell, 0705 292322

1. Suppose $(W(t))_{t\geq 0}$ is a one-dimensional standard Brownian motion and

$$X(t) = W^3(t) - 3tW(t),\ t \geq 0.$$

Find a stochastic process $(\Gamma(t))_{t\geq 0}$ which is adapted to the filtration generated by the Brownian motion such that

$$X(t) = \int_0^t \Gamma(s)dW(s),\ t \geq 0.$$

Solution. By the Itô lemma and product rule,

$$dX(t) = 3W^2(t)dW(t) + 3W(t)dt - 3W(t)dt - 3tdW(t)$$

$$= 3(W^2(t) - t)dW(t).$$

Thus if

$$\Gamma(t) = 3(W^2(t) - t)$$

we have

$$X(t) = \int_0^t \Gamma(s)dW(s),\ t \geq 0,$$

since $X(0) = 0$.

2. Let $W = (W(t))_{0\leq t\leq T}$ be a one-dimensional standard Brownian motion and $(\mathcal{F}(t))_{0\leq t\leq T}$ a filtration for W. Moreover, suppose the process $(X(t))_{0\leq t\leq T}$ solves the stochastic differential equation

$$dX(t) = \alpha dt + \sigma dW(t),\ 0 \leq t \leq T,$$

with the initial condition $X(0) = x_0$, where $\alpha, x_0 \in \mathbf{R}$ and $\sigma > 0$ are known parameters. Find a function $f(t, x)$ such that the random variable

$$E\left[e^{-\int_t^T X(u)du} \mid \mathcal{F}(t)\right]$$

is equal to $f(t, X(t))$ for every $t \in [0, T]$.

Solution. Suppose $\tau = T - t$ and $X(0) = x_0$. We have

$$X(t) = x_0 + \alpha t + \sigma W(t)$$

and

$$E\left[e^{-\int_t^T X(u)du} \mid \mathcal{F}(t)\right]$$
$$= e^{-x_0\tau - \frac{\alpha\tau}{2}(T+t)} E\left[e^{-\sigma\int_t^T W(u)du} \mid \mathcal{F}(t)\right]$$
$$= e^{-x_0\tau - \frac{\alpha\tau}{2}(T+t) - \sigma\tau W(t)} E\left[e^{-\sigma\int_t^T (W(u)-W(t))du} \mid \mathcal{F}(t)\right]$$
$$= e^{-x_0\tau - \frac{\alpha\tau}{2}(T+t) - \sigma\tau W(t)} E\left[e^{-\sigma\int_0^\tau W(u)du}\right].$$

Here

$$\int_0^\tau W(u)du \in N(0, \frac{\tau^3}{3})$$

as

$$\tilde{E}\left[\int_0^\tau W(u)du\right] = 0$$

and

$$E\left[(\int_0^\tau W(u)du)^2\right] = E\left[\int_0^\tau\int_0^\tau W(u)W(v)dudv\right]$$
$$= \int_0^\tau\int_0^\tau \min(u, v)dudv = \frac{\tau^3}{3}.$$

Thus

$$E\left[e^{-\int_t^T X(u)du} \mid \mathcal{F}(t)\right] = e^{-x_0\tau - \frac{\alpha\tau}{2}(T+t) - \sigma\tau W(t) + \frac{\sigma^2\tau^3}{6}}$$
$$= e^{-x_0\tau - \frac{\alpha\tau}{2}(T+t) - \tau(X(t) - x_0 - \alpha t) + \frac{\sigma^2\tau^3}{6}}$$
$$= e^{-\tau X(t) - \frac{\alpha\tau^2}{2} + \frac{\sigma^2\tau^3}{6}}.$$

Now

$$f(t,x) = e^{-x\tau - \frac{\alpha\tau^2}{2} + \frac{\sigma^2\tau^3}{6}}$$

$$= e^{-x(T-t) - \frac{\alpha}{2}(T-t)^2 + \frac{\sigma^2}{6}(T-t)^3}.$$

3. (Black-Scholes model with m stocks). Let $T > 0$ and set

$$X = \max_{0 \le t \le T} \left(\prod_{i=1}^{m} S_i(t) \right).$$

Moreover, let K be a given positive number and consider a European-style derivative paying the amount Y at time of maturity T, where

$$Y = \begin{cases} 1 \text{ if } X > K, \\ 0 \text{ if } X \le K. \end{cases}$$

Find the time zero price $\Pi_Y(0)$ of the derivative.

(Hint: If $\alpha \in \mathbf{R}$, $\beta, x > 0$, and W is a one-dimensional standard Brownian motion, then

$$P\left[\max_{0 \le t \le T} (\alpha t + \beta W(t)) \le x \right] = \Phi\left(\frac{x - \alpha T}{\beta\sqrt{T}}\right) - e^{\frac{2\alpha x}{\beta^2}} \Phi\left(-\frac{x + \alpha T}{\beta\sqrt{T}}\right)$$

where Φ is the cumulative distribution function of a Gaussian random variable with expectation 0 and variance 1.)

Solution. For each $i = 1, .., m$, let σ_i be the i:th row of the volatility matrix σ. Then

$$S_i(t) = S_i(0) e^{(r - \frac{|\sigma_i|^2}{2})t + \sigma_i \tilde{W}(t)}$$

and

$$X = A \exp\left(\max_{0 \le t \le T} \left((mr - \frac{1}{2} \sum_{i=1}^{m} \mid \sigma_i \mid^2) t + (\sum_{i=1}^{m} \sigma_i) \tilde{W}(t) \right) \right)$$

where

$$A = \prod_{i=1}^{m} S_i(0).$$

Hence

$$\Pi_Y(0) = e^{-rT}\tilde{E}[Y]$$
$$= e^{-rT}\tilde{P}[X > K]$$
$$= e^{-rT}\tilde{P}\left[\max_{0\le t\le T}\left((mr - \frac{1}{2}\sum_{i=1}^{m} \mid \sigma_i \mid^2)t + (\sum_{i=1}^{m}\sigma_i)\tilde{W}(t)\right) > \ln\frac{K}{A}\right].$$

Here under $\tilde{P}$ the process

$$\frac{\sum_{i=1}^{m}\sigma_i}{\mid \sum_{i=1}^{m}\sigma_i \mid}\tilde{W}(t),\ \ t \ge 0,$$

is a one-dimensional standard Brownian motion. Thus, if

$$\alpha = mr - \frac{1}{2}\sum_{i=1}^{m} \mid \sigma_i \mid^2$$

and

$$\beta = \mid \sum_{i=1}^{m}\sigma_i \mid$$

we have

$$\Pi_Y(0) = e^{-rT}(1 - \tilde{P}[X \le K])$$
$$= e^{-rT}\left(1 - \Phi(\frac{\ln\frac{K}{A} - \alpha T}{\beta\sqrt{T}}) + e^{\frac{2\alpha\ln\frac{K}{A}}{\beta^2}}\Phi(-\frac{\ln\frac{K}{A} + \alpha T}{\beta\sqrt{T}})\right)$$
$$= e^{-rT}\left(\Phi(-\frac{\ln\frac{K}{A} - \alpha T}{\beta\sqrt{T}}) + (\frac{K}{A})^{\frac{2\alpha}{\beta^2}}\Phi(-\frac{\ln\frac{K}{A} + \alpha T}{\beta\sqrt{T}})\right).$$

4. Let W be a one-dimensional standard Brownian motion and set

$$Q = \sum_{i=0}^{n-1}(W(t_{i+1}) - W(t_i))^2$$

where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T < \infty$. Show that $E[Q] = T$ and

$$\mathrm{Var}(Q) \le 2T\max_{0\le i\le n-1}(t_{i+1} - t_i).$$

5. Let $S(t)$ and $N(t)$ be the prices of two assets denominated in a common currency and let $\sigma(t) = (\sigma_1(t), ..., \sigma_d(t))$ and $\nu(t) = (\nu_1(t), ..., \nu_d(t))$ denote their respective volatility processes:

$$\begin{cases} d(D(t)S(t)) = D(t)S(t)\sigma(t) \cdot d\tilde{W}(t), \\ d(D(t)N(t)) = D(t)N(t)\nu(t) \cdot d\tilde{W}(t). \end{cases}$$

Suppose $N(t) > 0$ if $0 \leq t \leq T$, and take $N(t)$ as the numéraire. Define $\tilde{P}^{(N)}$ and $\tilde{W}^{(N)}$ and show that

$$dS^{(N)}(t) = S^{(N)}(t)\,[\sigma(t) - \nu(t)] \cdot d\tilde{W}^{(N)}(t)$$

where

$$S^{(N)}(t) = \frac{S(t)}{N(t)}, \ 0 \leq t \leq T.$$