SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

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No aids.
Questions on the exam: Christer Borell, 0705 292322

1. (Black-Scholes model, 2 stocks). Suppose

$$\frac{dS_j(t)}{S_j(t)}\frac{dS_k(t)}{S_k(t)} = \sum_{\nu=1}^{2} \sigma_{j\nu}\sigma_{k\nu},\ \ j,k = 1,2,$$

where

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

is a volatility matrix.

A European-style derivative pays the amount Y at time of maturity, where

$$Y = (S_2(T) - S_1(T))^2.$$

Find the price $\Pi_Y(0)$ of the derivative at time 0.

Solution. Using standard notation and the convention $\sigma_j = [\sigma_{j1}\ \sigma_{j2}] = (\sigma_{j1}, \sigma_{j2})$,

$$S_j(t) = S_j(0)e^{(r-\frac{|\sigma_j|^2}{2})t+\sigma_j\tilde{W}(t)}$$

and

$$\Pi_Y(0) = e^{-rT}\tilde{E}\left[(S_2(T) - S_1(T))^2\right]$$
$$= e^{-rT}\tilde{E}\left[S_2^2(T) - 2S_2(T)S_1(T) + S_1^2(T)\right].$$

Here

$$\tilde{E}\left[(S_j^2(T)\right] = \tilde{E}\left[S_j^2(0)e^{(2r-|\sigma_j|^2)T+2\sigma_j\tilde{W}(T)}\right]$$
$$= S_j^2(0)e^{(2r+|\sigma_j|^2)T}$$

and

$$\tilde{E}\left[S_2(T)S_1(T)\right] = \tilde{E}\left[S_2(0)S_1(0)e^{(2r-\frac{|\sigma_2|^2+|\sigma_1|^2}{2})T+(\sigma_2+\sigma_1)\tilde{W}(T)}\right]$$

$$= S_2(0)S_1(0)e^{(2r-\frac{|\sigma_2|^2+|\sigma_1|^2}{2})T+\frac{|\sigma_2+\sigma_1|^2}{2}T}$$

$$= S_2(0)S_1(0)e^{(2r+\sigma_2\cdot\sigma_1)T.}$$

Hence

$$\Pi_Y(0) = e^{rT}(S_2^2(0)e^{|\sigma_2|^2T} - 2S_2(0)S_1(0)e^{\sigma_2\cdot\sigma_1 T} + S_1^2(0)e^{|\sigma_1|^2T}).$$

2. A random variable X has the cumulative distribution function $\Phi^2(x)$, where $\Phi(x)$ is the cumulative distribution function of a Gaussian random variable with expectation 0 and variance 1. Find the moment-generating function of X.

Solution. If $\varphi(x) = \Phi'(x)$,

$$E\left[e^{tX}\right] = 2\int_{-\infty}^{\infty} e^{tx}\varphi(x)\Phi(x)dx$$

$$= 2e^{\frac{t^2}{2}}\int_{-\infty}^{\infty}\varphi(x-t)\Phi(x)dx = 2e^{\frac{t^2}{2}}\int_{-\infty}^{\infty}\varphi(x)\Phi(t-x)dx.$$

Set

$$f(t) = \int_{-\infty}^{\infty}\varphi(x)\Phi(t-x)dx,\ t\in\mathbf{R}.$$

Then

$$f'(t) = \int_{-\infty}^{\infty}\varphi(x)\varphi(t-x)dx$$

is the probability density function of the sum of two independent $N(0,1)$ distributed random variables and, hence

$$f'(t) = \frac{1}{2\sqrt{\pi}}e^{-\frac{t^2}{4}}.$$

Since $f(0) = \frac{1}{2}$ we have

$$f(t) = \Phi(\frac{t}{\sqrt{2}})$$

and

$$E\left[e^{tX}\right] = 2e^{\frac{t^2}{2}}\Phi(\frac{t}{\sqrt{2}}).$$

3. (Black-Scholes model) Suppose $a, b \in \mathbf{R}$, $b > a > 0$ and

$$X = \max_{0 \le u \le T} S(u).$$

A European-style derivative pays the amount Y at time of maturity T, where

$$Y = \begin{cases} 1 \text{ if } a \le X \le b \\ -1 \text{ otherwise }. \end{cases}$$

Find the price $\Pi_Y(t)$ of the derivative at time $t \in [0, T[$.

(Hint: If $\alpha \in \mathbf{R}$, $\beta, x > 0$, and W is a one-dimensional standard Brownian motion, then

$$P\left[\max_{0 \le t \le T}(\alpha t + \beta W(t)) \le x\right] = \Phi(\frac{x - \alpha T}{\beta\sqrt{T}}) - e^{\frac{2\alpha x}{\beta^2}}\Phi(-\frac{x + \alpha T}{\beta\sqrt{T}})$$

where Φ is as in Problem 2.)

Solution. To simplify notation we put $1_A = I(A)$. Moreover, let $\tau = T - t$, $U = I(X < a)$, and $V = I(X \le b)$. Then

$$Y = 2(V - U) - 1$$

and

$$\Pi_Y(t) = 2(\Pi_V(t) - \Pi_U(t)) - e^{-r\tau}$$

$$= 2e^{-r\tau}\left(\tilde{E}\left[I(\max_{0 \le u \le T} S(u) \le b) \mid \mathcal{F}(t)\right] - \tilde{E}\left[I(\max_{0 \le u \le T} S(u) < a) \mid \mathcal{F}(t)\right]\right) - e^{-r\tau}.$$

If $u \ge t$,

$$S(u) = S(t)e^{(r - \frac{\sigma^2}{2})(u-t) + \sigma(\tilde{W}(u) - \tilde{W}(t)}$$

and

$$\max_{0 \le u \le T} S(u) = \max(\max_{0 \le u \le t} S(u), \max_{t \le u \le T} S(u))$$

$$= \max(\max_{0 \le u \le t} S(u), S(t)e^{\max_{t \le u \le T}((r - \frac{\sigma^2}{2})(u-t) + \sigma(\tilde{W}(u) - \tilde{W}(t)))}).$$

Hence

$$\tilde{E}\left[I(\max_{0\le u\le T} S(u) < a \mid \mathcal{F}(t)\right]$$

$$= \tilde{E}\left[I(\max(\max_{0\le u\le t} S(u), S(t)e^{\max_{t\le u\le T}((r-\frac{\sigma^2}{2})(u-t)+\sigma(\tilde{W}(u)-\tilde{W}(t)))}) < a) \mid \mathcal{F}(t)\right]$$

$$= \tilde{E}\left[I(\max(m, se^{\max_{t\le u\le T}((r-\frac{\sigma^2}{2})(u-t)+\sigma(\tilde{W}(u)-\tilde{W}(t)))}) < a)\right]_{|(m,s)=(\max_{0\le u\le t} S(u),S(t))}$$

$$= \tilde{E}\left[I(\max(m, se^{\max_{0\le u\le \tau}((r-\frac{\sigma^2}{2})u+\sigma\tilde{W}(u))}) < a)\right]_{|(m,s)=(\max_{0\le u\le t} S(u),S(t))}.$$

Now if $m < a$,

$$E\left[I(\max(m, se^{\max_{0\le u\le \tau}((r-\frac{\sigma^2}{2})u+\sigma W(u))})) < a)\right]$$

$$= P\left[se^{\max_{0\le u\le \tau}((r-\frac{\sigma^2}{2})u+\sigma W(u))} < a\right]$$

$$= P\left[\max_{0\le u\le \tau}((r-\frac{\sigma^2}{2})u+\sigma W(u)) < \ln\frac{a}{s})\right]$$

$$= \Phi(\frac{\ln\frac{a}{s}-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - e^{\frac{2(r-\frac{\sigma^2}{2})\ln\frac{a}{s}}{\sigma^2}}\Phi(-\frac{\ln\frac{a}{s}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}).$$

Hence

$$\tilde{E}\left[I(\max_{0\le u\le T} S(u) < a \mid \mathcal{F}(t)\right]$$

$$= 1_{[0,a[}(m)\left(\Phi(\frac{\ln\frac{a}{s}-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - (\frac{a}{s})^{\frac{2r}{\sigma^2}-1}\Phi(-\frac{\ln\frac{a}{s}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\right).$$

In a similar way, it $m \le b$,

$$\tilde{E}\left[I(\max_{0\le u\le T} S(u) \le b \mid \mathcal{F}(t)\right]$$

$$= 1_{[0,b]}(m)\left(\Phi(\frac{\ln\frac{b}{s}-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - (\frac{b}{s})^{\frac{2r}{\sigma^2}-1}\Phi(-\frac{\ln\frac{b}{s}+(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\right).$$

Thus if

$$m = \max_{0\le u\le t} S(u)$$

we have

$$\Pi_Y(t) = 2e^{-r\tau}\left(\tilde{E}\left[I(\max_{0\le u\le T} S(u) \le b) \mid \mathcal{F}(t)\right] - \tilde{E}\left[I(\max_{0\le u\le T} S(u) < a) \mid \mathcal{F}(t)\right]\right) - e^{-r\tau}$$

$$= 2e^{-r\tau}1_{[0,b]}(\max_{0\le u\le t} S(u))\left(\Phi(\frac{\ln\frac{b}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - (\frac{b}{S(t)})^{\frac{2r}{\sigma^2}-1}\Phi(-\frac{\ln\frac{b}{S(t)} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\right)$$

$$-2e^{-r\tau}1_{[0,a[}(\max_{0\le u\le t} S(u))\left(\Phi(\frac{\ln\frac{a}{S(t)} - (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}) - (\frac{a}{S(t)})^{\frac{2r}{\sigma^2}-1}\Phi(-\frac{\ln\frac{a}{S(t)} + (r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}})\right)$$

$$-e^{-r\tau}.$$

4. Let $W = (W(t))_{t\ge 0}$ be a one-dimensional Brownian motion and $(\mathcal{F}(t))_{t\ge 0}$ a filtration for W. (a) Prove that W is a martingale. (b) Suppose $\sigma \in \mathbf{R}$. Prove that the process $Z(t) = \exp\left\{\sigma W(t) - \frac{\sigma^2}{2}t\right\}$, $t \ge 0$, is a martingale.

5. Let W be a one-dimensional standard Brownian motion and suppose $m > 0$ and $\tau_m = \min\{t > 0;\ W(t) = m\}$. Use the formula

$$P[\tau_m \le t,\ W(t) \le w] = P[W(t) \ge 2m - w],\ w \le m,$$

to prove that τ_m has the density

$$f_m(t) = \frac{m}{t\sqrt{2\pi t}}e^{-\frac{m^2}{2t}},\ t > 0.$$