SOLUTIONS
FINANCIAL DERIVATIVES AND STOCHASTIC ANALYSIS
(CTH[$tma285$], GU[$MMA710$])

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No aids.
Questions on the exam: Jacob Leander 0703-088304

1. Let $W = (W_1(t), W_2(t))_{0\le t\le T}$ be a 2-dimensional standard Brownian motion. Show that the process $(e^{W_1(t)} \cos W_2(t))_{0\le t\le T}$ is a martingale.

Solution. We have

$$d(e^{W_1(t)} \cos W_2(t))) = (de^{W_1(t)}) \cos W_2(t)) + e^{W_1(t)}(d \cos W_2(t)) + (de^{W_1(t)})(d \cos W_2(t))$$

$$= e^{W_1(t)}(dW_1(t) + \frac{1}{2}dt) \cos W_2(t) + e^{W_1(t)}(-\sin W_2(t)dW_2(t) - \frac{1}{2} \cos W_2(t)dt) + 0$$

$$= e^{W_1(t)}dW_1(t) - e^{W_1(t)} \sin W_2(t)dW_2(t).$$

Thus, if $\Gamma(t) = e^{W_1(t)}(1, -\sin W_2(t))$, then

$$e^{W_1(t)} \cos W_2(t)) = 1 + \int_0^t \Gamma(u) \cdot dW(u)$$

and it follows that $(e^{W_1(t)} \cos W_2(t))_{0\le t\le T}$ is a martingale.

2. (Black-Scholes model, 2 stocks). For each $j \in \{1, 2\}$, suppose

$$dS_j(t) = S_j(t)(\alpha_j dt + \sigma_{j1}dW_1(t) + \sigma_{j2}dW_2(t)),$$

where $W = (W_1, W_2)$ is a 2-dimensional standard Brownian motion, $\alpha_1, \alpha_2 \in \mathbf{R}$, and

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

is a volatility matrix. Find $\mathrm{Cov}(S_1(t), \ln S_2(t))$.

Solution. We have

$$S_j(t) = S_j(0)e^{\beta_j t+\sigma_{j1}W_1(t)+\ \sigma_{j2}W_2(t)}$$

where

$$\beta_j = \alpha_j - \frac{1}{2}(\sigma_{j1}^2 + \sigma_{j2}^2) \text{ if } j = 1, 2.$$

Now

$$\begin{aligned}
&\mathrm{Cov}(S_1(t), \ln S_2(t)) \\
&= \mathrm{Cov}(S_1(0)e^{\beta_1 t+\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}, \ln S_2(0) + \beta_2 t + \sigma_{21}W_1(t) +\ \sigma_{22}W_2(t)) \\
&= \mathrm{Cov}(S_1(0)e^{\beta_1 t+\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}, \sigma_{21}W_1(t) +\ \sigma_{22}W_2(t)) \\
&= S_1(0)e^{\beta_1 t}\mathrm{Cov}(e^{\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}, \sigma_{21}W_1(t) +\ \sigma_{22}W_2(t)) \\
&= S_1(0)e^{\beta_1 t}E\left[e^{\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}(\sigma_{21}W_1(t) +\ \sigma_{22}W_2(t))\right].
\end{aligned}$$

Here

$$\begin{aligned}
&E\left[e^{\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}(\sigma_{21}W_1(t) +\ \sigma_{22}W_2(t))\right] \\
&= \sigma_{21}E\left[e^{\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}W_1(t)\right] +\ \sigma_{22}E\left[e^{\sigma_{11}W_1(t)+\ \sigma_{12}W_2(t)}W_2(t)\right] \\
&= \sigma_{21}E\left[e^{\sigma_{11}W_1(t)}W_1(t)\right]E\left[e^{\sigma_{12}W_2(t)}\right] +\ \sigma_{22}E\left[e^{\sigma_{11}W_1(t)}\right]E\left[e^{\sigma_{12}W_2(t)}W_2(t)\right].
\end{aligned}$$

Moreover, if $G \in N(0, 1)$ and u is a real number,

$$E\left[e^{uG}G\right] = ue^{\frac{u^2}{2}}$$

and we get

$$\begin{aligned}
&\sigma_{21}E\left[e^{\sigma_{11}W_1(t)}W_1(t)\right]E\left[e^{\sigma_{12}W_2(t)}\right] +\ \sigma_{22}E\left[e^{\sigma_{11}W_1(t)}\right]E\left[e^{\sigma_{12}W_2(t)}W_2(t)\right] \\
&= (\sigma_{11}\sigma_{21} + \sigma_{12}\ \sigma_{22})te^{\frac{t}{2}(\sigma_{11}^2+\sigma_{12}^2)}.
\end{aligned}$$

From the above

$$\begin{aligned}
&\mathrm{Cov}(S_1(t), \ln S_2(t)) \\
&= S_1(0)(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})te^{\alpha_1 t}.
\end{aligned}$$

3. Let $S_f(t)$ denote the price in foreign currency of a foreign stock and $Q(t)$ the exchange rate, which gives units of domestic currency per unit of foreign currency. Moreover, consider a European-style derivative which pays the amount $Y = S_f(T)$ in domestic currency at time of maturity T. Derive the price $\Pi_Y(0)$ of the derivative in domestic currency at time 0 under the following assumptions:

(*i*) The domestic and foreign interest rates are constant and denoted by r and r_f, respectively.

(*ii*) There exists a 2-dimensional standard Brownian motion $W = (W_1, W_2)$ such that

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f} dt + \sigma_{11} dW_1(t) + \sigma_{12} dW_2(t)), \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t)), \end{cases}$$

where $\alpha_{S_f}, \alpha_Q, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \in \mathbf{R}$ and the matrice $(\sigma_{ik})_{1 \leq i,k \leq 2}$ is invertible.

Solution. If $\sigma_i = (\sigma_{i1}, \sigma_{i2})$, $i = 1, 2$,

$$\begin{cases} dS_f(t) = S_f(t)(\alpha_{S_f} dt + \sigma_1 \cdot dW(t)), \\ dQ(t) = Q(t)(\alpha_Q dt + \sigma_2 \cdot dW(t)). \end{cases}$$

Let $B_f(t) = B_f(0)e^{r_f t}$ be the foreign bond price at time t and

$$\begin{cases} S(t) = Q(t)S_f(t) \text{ if } 0 \leq t \leq T, \\ U(t) = Q(t)B_f(t) \text{ if } 0 \leq t \leq T. \end{cases}$$

Now $(S(t))_{0 \leq t \leq T}$ and $(U(t))_{0 \leq t \leq T}$ can be viewed as price processes of domestic assets and

$$\begin{cases} dS(t) = S(t)((\cdot)dt + (\sigma_1 + \sigma_2) \cdot dW(t)), \\ dU(t) = U(t)((\cdot)dt + \sigma_2 \cdot dW(t)), \end{cases}$$

for appropriate drift coefficients which need not be specified. If $\tilde{P}$ denotes the domestic risk-neutral measure we have

$$\begin{cases} dS(t) = S(t)(rdt + (\sigma_1 + \sigma_2) \cdot d\tilde{W}(t)), \\ dU(t) = U(t)(rdt + \sigma_2 \cdot d\tilde{W}(t)), \end{cases}$$

where under $\tilde{P}$ the process $\tilde{W}$ is a 2-dimensional standard Brownian motion. Moreover,

$$Y = B_f(T)\frac{Q(T)S_f(T)}{Q(T)B_f(T)} = B_f(T)\frac{S(T)}{U(T)}.$$

Here

$$\frac{S(T)}{U(T)} = \frac{S_f(0)}{B_f(0)} e^{\frac{1}{2}(|\sigma_2|^2 - |\sigma_1+\sigma_2|^2)T + \sigma_1 \cdot \tilde{W}(T)}$$

and we get

$$\Pi_Y(0) = B_f(T)\frac{S_f(0)}{B_f(0)} e^{-rT}\tilde{E}\left[e^{\frac{1}{2}(|\sigma_2|^2 - |\sigma_1+\sigma_2|^2)T + \sigma_1 \cdot \tilde{W}(T)}\right]$$

$$= S_f(0)e^{(r_f - r - \sigma_1 \cdot \sigma_2)T}.$$

4. (Cox-Ingersoll-Ross interest rate model) Suppose $R(0) = r$ and

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t), \ t \geq 0,$$

where $\alpha, \beta, \sigma > 0$ and $r \in \mathbf{R}$ are parameters and W is a standard 1-dimensional Brownian motion. Prove that

$$E\left[R(t)\right] = e^{-\beta t}r + \frac{\alpha}{\beta}(1 - e^{-\beta t}).$$

5. Consider a market model possessing a unique risk-neutral measure $\tilde{P}$. Using standard notation, $B(t,T) = \frac{1}{D(t)}\tilde{E}\left[D(T) \mid \mathcal{F}(t)\right]$, $\mathrm{For}_S(t,T) = \frac{S(t)}{B(t,T)}$, and $\mathrm{Fut}_S(t,T) = \tilde{E}\left[S(T) \mid \mathcal{F}(t)\right]$. (a) Show that the forward-futures spread equals

$$\mathrm{For}_S(0,T) - \mathrm{Fut}_S(0,T) = \frac{\tilde{C}}{B(0,T)},$$

where $\tilde{C}$ is the covariance of $D(T)$ and $S(T)$ under the risk-neutral measure $\tilde{P}$. (b) Prove that $\mathrm{For}_S(0,T) = \mathrm{Fut}_S(0,T)$ if the discount process $(D(t))_{0 \leq t \leq T}$ is nonrandom.