Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) August 15th$^{\text{th}}$, 2017

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Remark: (1) No aids permitted (2) This exam consists of two pages! (turn the page)

1. Consider the 1+1 dimensional market

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{1}$$

where we assume that the market parameters $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times.

   (a) Give and explain the definition of risk-neutral probability measure of the market (max 1 point).

   (b) Assume that the price $\{\Pi_Y(t)\}_{t\in[0,T]}$ of the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is given by the risk-neutral pricing formula. Show there exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$, such that

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]$$

   where $\Pi_Y^*(t)$ is the discounted value of the derivative (max. 2 points).

   (b) Show that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

   is self-financing and hedging the derivative (max. 2 points).

2. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant. The Asian call with geometric average is the European style derivative with pay-off

$$Y = \left(\exp\left(\frac{1}{T}\int_0^T \log S(t)\,dt\right) - K\right)_+,$$

where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price of this option (max. 3 points) and the corresponding put-call parity (max. 2 points).

3. Assume that $R(t) = r > 0$ is a deterministic constant. A variance swap is a forward contract with pay-off

$$Y = \frac{\kappa}{T} \int_0^T \sigma^2(t)\, dt - K,$$

where κ is the number of trading days in one year. The value K_* of K such that the initial value of the swap is zero is called the swap strike. Compute K_* assuming the CEV model $\sigma(t) = \sigma_0 \sqrt{S(t)}$ for the instantaneous volatility, where $\sigma_0 > 0$ is a constant (max. 5 points)