Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) March 14th, 2017

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Remark: (1) No aids permitted (2) This exam consists of 2 pages! (turn the page)

1. Consider the 1+1 dimensional market

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{1}$$

where we assume that the market parameters $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times.

(a) Give and explain the definition of risk-neutral probability measure of the market (max 1 point).

(b) Assume that the price $\{\Pi_Y(t)\}_{t\in[0,T]}$ of the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is given by the risk-neutral pricing formula. Show there exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$, such that

$$\Pi^*_Y(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]$$

where $\Pi^*_Y(t)$ is the discounted value of the derivative (max. 2 points).

(b) Show that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is self-financing and hedging the derivative (max. 2 points).

2. Assume that the market parameters in (1) are constants. The Asian call, resp. put, with maturity $T > 0$ and strike $K > 0$ is the derivative with pay off

$$Y_{\text{call}} = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)_+ \quad \text{resp.} \quad Y_{\text{put}} = \left(K - \frac{1}{T}\int_0^T S(t)\,dt\right)_+.$$

Derive the partial differential equation and the terminal condition for the pricing function of the Asian call/put (max. 2 points). Show that the prices $\Pi_{\text{call}}(t)$, $\Pi_{\text{put}}(t)$ of the Asian call/put satisfy the parity identity

$$\Pi_{\text{call}}(t) - \Pi_{\text{put}}(t) = e^{-r(T-t)}\left[\frac{1}{T}Y(t) + \frac{e^{r(T-t)}-1}{rT}S(t) - K\right], \quad t \in [0,T]$$

where $Y(t) = \int_0^t S(\tau)\,d\tau$ (max. 3 points).

3. Assume that the spot interest rate in the risk-neutral probability is given by the *Hull-White* model:

$$dR(t) = (a - bR(t))\,dt + c d\widetilde{W}(t),$$

where a, b, c are constants. Derive the partial differential equation and the terminal condition satisfied by the pricing function $f(t, x)$ of the zero-coupon bond with maturity T and face value 1 (max. 1 point). Find $f(t, x)$ (max. 1 point). HINT: Use the ansatz $f(t, x) = \exp(-xA(T-t) - G(T-t))$, where A, G are deterministic functions of time. Finally, use the HJM method to derive the dynamics of the instantaneous forward rate $F(t, T)$ in the physical probability (max 3 points).