Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) August 31$^{\text{st}}$, 2018

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Remark: (1) No aids permitted (2) This exam consists of 2 pages! (turn the page)

1. Consider the 1+1 dimensional market

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{1}$$

where we assume that the market parameters $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times.

(a) Give and explain the definition of risk-neutral probability measure of the market (max 1 point).

(b) Assume that the price $\{\Pi_Y(t)\}_{t\in[0,T]}$ of the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is given by the risk-neutral pricing formula. Show there exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$, such that

$$\Pi_Y^*(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]$$

where $\Pi_Y^*(t)$ is the discounted value of the derivative (max. 2 points).

(b) Show that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is self-financing and hedging the derivative (max. 2 points).

2. Assume that the spot interest rate in the risk-neutral probability is given by the *Hull-White* model:

$$dR(t) = (a - bR(t))\,dt + cd\widetilde{W}(t),$$

where a, b, c are constants. Derive the partial differential equation and the terminal condition satisfied by the pricing function $f(t,x)$ of the zero-coupon bond with maturity T and face value 1 (max. 1 point). Find $f(t,x)$ (max. 1 point). HINT: Use the ansatz $f(t,x) = \exp(-xA(T-t) - G(T-t))$, where A, G are deterministic functions of time. Finally, use the HJM method to derive the dynamics of the instantaneous forward rate $F(t,T)$ in the physical probability (max 3 points).

3. Let $S_1(t), S_2(t)$ be the prices at time t of two stocks. A two asset correlation call option with strikes K_1, K_2 and maturity T is a European style derivative with pay-off

$$Y = \begin{cases} \max\{S_2(T) - K_2, 0\} & \text{if } S_1(T) > K_1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that the risk-free rate is a constant r and that the prices of the stock are given by the SDEs

$$dS_1(t) = rS_1(t) + \sigma_{11}S_1(t)d\widetilde{W}_1(t) + \sigma_{12}S_1(t)d\widetilde{W}_2(t) \tag{2}$$

$$dS_2(t) = rS_2(t) + \sigma_{21}S_2(t)d\widetilde{W}_1(t) + \sigma_{22}S_2(t)d\widetilde{W}_2(t) \tag{3}$$

where $\{\widetilde{W}_1(t)\}_{t\geq 0}, \{\widetilde{W}_1(t)\}_{t\geq 0}$ are independent Brownian motion in the risk neutral probability measure.

Derive the PDE and the terminal condition satisfied by the Black-Scholes pricing function of this derivative (max. 2 points). Derive a formula for the Black-Scholes pricing function of the option in the special case when the stock prices are independent (max. 3 points).