Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) June 8th$^{\text{th}}$, 2018

Questions on the exam: Olof Elias (5325)

Remark: (1) No aids permitted

1. Consider the 1+1 dimensional market

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{1}$$

where we assume that the market parameters $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times.

(a) Give and explain the definition of risk-neutral probability measure of the market (max 1 point).

(b) Assume that the price $\{\Pi_Y(t)\}_{t\in[0,T]}$ of the European derivative on the stock with pay-off Y and time of maturity $T > 0$ is given by the risk-neutral pricing formula. Show there exists a stochastic process $\{\Delta(t)\}_{t\in[0,T]}$, adapted to $\{\mathcal{F}_W(t)\}_{t\in[0,T]}$, such that

$$\Pi^*_Y(t) = \Pi_Y(0) + \int_0^t \Delta(s)d\widetilde{W}(s), \quad t \in [0,T]$$

where $\Pi^*_Y(t)$ is the discounted value of the derivative (max. 2 points).

(b) Show that the portfolio $\{h_S(t), h_B(t)\}_{t\in[0,T]}$ given by

$$h_S(t) = \frac{\Delta(t)}{D(t)\sigma(t)S(t)}, \quad h_B(t) = (\Pi_Y(t) - h_S(t)S(t))/B(t)$$

is self-financing and hedging the derivative (max. 2 points).

2. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is constant. The Asian call with geometric average is the European style derivative with pay-off

$$Y = \left(\exp\Big(\frac{1}{T}\int_0^T \log S(t)\,dt\Big) - K\right)_+,$$

where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price at time $t = 0$ of this option (max. 4 points) and the corresponding put-call parity (max. 1 point). HINT: $\int_0^t W(s)\, ds$ is normally distributed, for all $t > 0$.

3. Consider the CIR model for the interest rate process $\{R(t)\}_{t\geq 0}$ of a bond:

$$dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}d\widetilde{W}(t)$$

where a, b, σ are positive constants and $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk neutral measure. Derive the expectation value and the variance of $\{R(t)\}_{t\geq 0}$ (max. 3 points). Assuming that the value $B(t, T)$ at time $t < T$ of a zero-coupon bond expiring at time T has the form $B(t, T) = f(t, R(t))$, for a smooth function $f : [0, T] \times (0, \infty) \to (0, \infty)$, derive the partial differential equation satisfied by f (max. 2 points).