Exam for the course "Financial derivatives and PDE's" (CTH[*tma285*], GU[*mma711*]) March 13th$^{\text{th}}$, 2018

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Remark: (1) No aids permitted

1. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant.
   (a) Give and justify the definition for the fair price $\widetilde{\Pi}(0)$ at time $t = 0$ of the perpetual American put on the stock with strike K (max. 1 point).
   (b) Prove that $\widetilde{\Pi}(0) = v_{L_*}(S(0))$, where
   $$v_{L_*}(x) = \begin{cases} K - x, & 0 \leq x \leq L_* \\ (K - L_*)\big(\frac{x}{L_*}\big)^{-\frac{2r}{\sigma^2}}, & x > L_* \end{cases}, \qquad L_* = \frac{2r}{2r + \sigma^2}K.$$
   (max. 4 points).
2. Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is constant. The Asian call with geometric average is the European style derivative with pay-off
   $$Y = \Big(\exp\Big(\frac{1}{T}\int_0^T \log S(t)\,dt\Big) - K\Big)_+,$$
   where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price at time $t = 0$ of this option (max. 4 points) and the corresponding put-call parity (max. 1 point). HINT: $\int_0^t W(s)\,ds$ is normally distributed, for all $t > 0$.
3. Assume that the price $S(t)$ of a stock follows a generalized geometric Brownian motion with instantaneous volatility $\{\sigma(t)\}_{t\geq 0}$ given by the Heston model $d\sigma^2(t) = a(b - \sigma^2(t))\,dt + c\sigma(t)\,d\widetilde{W}(t)$, where $\{\widetilde{W}(t)\}_{t\geq 0}$ is a Brownian motion in the risk-neutral probability measure and a, b, c are constants such that $2ab \geq c^2 > 0$. Let also the

interest rate r be constant. A volatility call option with strike K and maturity T is a financial derivative with pay-off

$$Y = N \left(\sqrt{\frac{\kappa}{T} \int_0^T \sigma^2(t) \, dt} - K \right)_+ ,$$

where κ is the number of trading days in one year and N is the notional amount of the option. Find the partial differential equation and the terminal value satisfied by the pricing function of this derivative (max. 5 points).