Exam for the course “Financial derivatives and PDE's” (CTH[*tma285*], GU[*mma711*]) March 19$^{\text{th}}$, 2019

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Remark: (1) No aids permitted

1. Consider the 1+1 dimensional market

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad dB(t) = B(t)R(t)dt, \tag{1}$$

where we assume that the market parameters $\{\alpha(t)\}_{t\geq 0}$, $\{\sigma(t)\}_{t\geq 0}$, $\{R(t)\}_{t\geq 0}$ are adapted to $\{\mathcal{F}_W(t)\}_{t\geq 0}$ and that $\sigma(t) > 0$ almost surely for all times. Let also Y be a $\mathcal{F}_W(T)$-measurable random variable.

   (a) Give and explain the definition of risk-neutral price $\Pi_Y(t)$ of the European derivative with pay-off Y and maturity T (max. 1 point).

   (b) Show that $\{S^*(t)\}_{t\geq 0}$ and $\{\Pi_Y^*(t)\}_{t\geq 0}$ are martingales in the risk-neutral probability measure, where $X^*(t)$ is the discounted (at time $t = 0$) value of $X(t)$ (max. 2 points).

   (b) Derive the pricing PDE for the derivative assuming that (i) $Y = g(S(T))$, (ii) the interest rate of the money market is constant and (iii) the volatility of the stock is a deterministic function of the stock price (local volatility) (max. 2 points).

   Solution: See Lecture Notes

2. Assume that the spot interest rate in the risk-neutral probability is given by the *Ho-Lee* model:

$$dR(t) = \alpha(t)\, dt + \sigma d\widetilde{W}(t),$$

where $\sigma > 0$ is a constant and $\alpha(t)$ is a deterministic function of time. Derive the risk-neutral price $B(t, T)$ of the ZCB with face value 1 and maturity T (max. 2 points). Use the HJM method to derive the dynamics of the instantaneous forward rate $F(t, T)$ in the physical probability (max. 3 points).

Solution. Letting $B(t, T) = v(t, R(t))$, and imposing that the drift of $B^*(t, T)$ is zero, we find that $v(t, x)$ satisfies the PDE

$$\partial_t v + \alpha(t)\partial_x v + \frac{\sigma^2}{2}\partial_x^2 v = xv, \quad x > 0, t \in (0, T)$$

with the terminal condition $v(T, x) = 1$. Looking for solutions of the form $v(t, x) = e^{-xC(T,t)-A(T,t)}$ we find that C, A satisfy $C(T, T) = 0$, $A(T, T) = 0$, and

$$\partial_t C = -1, \quad \partial_t A = C^2 - C\alpha(t).$$

Solving the ODE's,

$$C(T,t) = (T-t), \quad A(T,t) = -\sigma^2 \frac{(T-t)^3}{6} + \int_t^T (T-s)\alpha(s)\,ds.$$

This completes the first part of the exercise (2 points). The forward rate is given by

$$F(t,T) = -\partial_T \log B(t,T) = R(t)\partial_T C(T,t) + \partial_T A(T,t),$$

hence

$$\begin{aligned} dF(t,T) &= dR(t)\partial_T C(T,t) + R(t)\partial_t\partial_T C(T,t)\,dt + \partial_t\partial_T A(T,t)\,dt \\ &= \sigma^2(T-t)\,dt + \sigma\, d\widetilde{W}(t) \end{aligned}$$

Now, the general form of the forward rate in the HJM approach in the risk-neutral probability measure is

$$dF(t,T) = \sigma(t,T)\overline{\sigma}(t,T)\,dt + \sigma(t,T)\,d\widetilde{W}(t), \quad \text{where} \quad \overline{\sigma}(t,T) = \int_t^T \sigma(t,v)\,dv.$$

Comparing with the expression above we find $\sigma(t,T) = \sigma$, so the dynamics of the forward rate in the physical probability is

$$\begin{aligned} dF(t,T) &= (\theta(t)\sigma(t,T) + \sigma(t,T)\overline{\sigma}(t,T))\,dt + \sigma(t,T)dW(t) \\ &= \sigma(\theta(t) + \sigma(T-t))\,dt + \sigma dW(t), \end{aligned}$$

where $\theta(t)$, the market price of risk, is any adapted process (chosen by calibrating the model). This completes the second part of the exercise (3 points).

3. Let $0 < s < T$ and assume that the market parameters are constant (Black-Scholes market). Find the risk-neutral price $\Pi_Y(t)$, $t \in [0,T]$, of the European derivative with pay-off $Y = (S(T) - S(s))_+$ and maturity T (max. 3 points). Find also the put-call parity relation satisfied by this derivative and the derivative with pay-off $Z = (S(s) - S(T))_+$ (max. 2 points).

   Solution. First we note that for $t \geq s$, we can consider the derivative as a standard call option with maturity T and *known* strike price $K = S(s)$, hence

$$\Pi_Y(t) = C(t, S(t), S(s), T), \quad \text{for } t \geq s,$$

   where $C(t,x,K,T)$ is the Black-Scholes price function of the standard call option with strike K and maturity T. For $t < s$ we write

$$\begin{aligned} \Pi_Y(t) &= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S(T)-S(s))_+ | \mathcal{F}_W(t)] \\ &= e^{-r(T-t)}\widetilde{\mathbb{E}}[S(t)(e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(\widetilde{W}(T)-\widetilde{W}(t))} - e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(\widetilde{W}(s)-\widetilde{W}(t))})_+ | \mathcal{F}_W(t)] \\ &= e^{-r(T-t)}S(t)\widetilde{\mathbb{E}}[(e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(\widetilde{W}(T)-\widetilde{W}(t))} - e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(\widetilde{W}(s)-\widetilde{W}(t))})_+] \end{aligned}$$

where in the last step we used that $S(t)$ is measurable with respect to $\mathcal{F}_W(t)$ and that the Brownian motion increments are independent of $\mathcal{F}_W(t)$. It follows that

$$\begin{aligned}\Pi_Y(t) &= e^{-r(T-t)} S(t) \widetilde{\mathbb{E}}[e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(\widetilde{W}(s)-\widetilde{W}(t))}(e^{(r-\frac{1}{2}\sigma^2)(T-s)+\sigma(\widetilde{W}(T)-\widetilde{W}(s))}-1)_+] \\ &= e^{-r(T-t)} S(t) \widetilde{\mathbb{E}}[e^{(r-\frac{1}{2}\sigma^2)(s-t)+\sigma(\widetilde{W}(s)-\widetilde{W}(t))}] \widetilde{\mathbb{E}}[(e^{(r-\frac{1}{2}\sigma^2)(T-s)+\sigma(\widetilde{W}(T)-\widetilde{W}(s))}-1)_+]\end{aligned}$$

where in the last step we used that the two Brownian motion increments are independent. Computing the expectations using that $W(t_2)-W(t_1) \in \mathcal{N}(0, t_2-t_1)$, for all $t_2 > t_1$, we find

$$\Pi_Y(t) = S(t)(\Phi(a+\sigma\sqrt{T-t}) - e^{-r(T-s)}\Phi(a)), \quad a = \frac{(r-\frac{1}{2}\sigma^2)\sqrt{T-s}}{\sigma}$$

This completes the solution to the first part of the exercise (3 points). As to the put call parity, let $Z = (S(s)-S(T))_+$. As $(x-y)_+ - (y-x)_+ = x-y$, we have

$$\begin{aligned}\Pi_Y(t) - \Pi_Z(t) &= e^{-r(T-t)}\widetilde{\mathbb{E}}[(S(T)-S(s))_+|\mathcal{F}_W(t)] - e^{-r(T-t)}\widetilde{\mathbb{E}}[(S(s)-S(T))_+|\mathcal{F}_W(t)] \\ &= e^{-r(T-t)}\widetilde{\mathbb{E}}[S(T)-S(s)|\mathcal{F}_W(t)] = e^{-r(T-t)}(\widetilde{\mathbb{E}}[S(T)|\mathcal{F}_W(t)] - \widetilde{\mathbb{E}}[S(s)|\mathcal{F}_W(t)]) \\ &= \begin{cases} S(t) - e^{-r(T-t)}S(s) & \text{if } s \le t \\ S(t) - e^{-r(T-s)}S(t) & \text{if } s > t. \end{cases}\end{aligned}$$

Hence the put-call parity is

$$\Pi_Y(t) - \Pi_Z(t) = S(t) - e^{-r(T-\max(s,t))}S(\min(s,t)).$$

This completes the answer to the second question (2 points)