Solutions to exercises in the exam

Remark: The other solutions can be found in Appendix 6.D of the lecture notes.

1 Asian call with geometric average

Assume that the price $S(t)$ of a stock follows a geometric Brownian motion with instantaneous volatility $\sigma > 0$ and that the risk-free interest rate r is a positive constant. The Asian call with geometric average is the European style derivative with pay-off

$$Y = \left(\exp\left(\frac{1}{T}\int_0^T \log S(t)\,dt\right) - K\right)_+,$$

where $T > 0$ and $K > 0$ are respectively the maturity and strike of the call. Derive an exact formula for the Black-Scholes price of this option at time $t = 0$ (max. 4 points) and the corresponding put-call parity (max. 1 point).

Solution. We have

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\widetilde{W}(t)}.$$

Hence

$$Y = \left(S(0)e^{(r-\sigma^2/2)T/2+\sigma X(T)/T} - K\right)_+, \quad X(T) = \int_0^T W(t)\,dt.$$

Recall that $X(T)$ is normally distributed. Moreover

$$\mathbb{E}[X(T)] = \int_0^T \mathbb{E}[W(t)]\,dt = 0$$

and

$$\mathrm{Var}[X(T)] = \mathbb{E}[X(T)^2] = \mathbb{E}[\int_0^T\int_0^T W(t)W(s)\,dt\,ds] = \int_0^T\int_0^T \mathbb{E}[W(t)W(s)]\,dt\,ds$$

If $s < t$, then $\mathbb{E}[W(t)W(s)] = \mathbb{E}[(W(t) - W(s))W(s)] + \mathbb{E}[W(s)^2] = \mathbb{E}[W(s)^2] = s$ and similarly $\mathbb{E}[W(t)W(s)] = t$ if $s > t$. Hence

$$\mathrm{Var}[X(T)] = \int_0^T\int_0^T \min(s,t)\,dt\,ds = 2\int_0^T\int_0^s t\,dt\,ds = \frac{T^3}{3}.$$

Thus $X(T) \in N(0, T^3/3)$ and so $G := X(T)/\sqrt{T^3/3} \in N(0,1)$. The pay-off of the derivative can be rewritten as

$$Y = \left(S(0)e^{(r-\sigma^2/2)T/2+\sigma\sqrt{T/3}G} - K\right)_+ = \left(S(0)e^{(\hat{r}-\hat{\sigma}^2/2)T+\hat{\sigma}\sqrt{T}G} - K\right)_+$$

where

$$\hat{\sigma} = \frac{\sigma}{\sqrt{3}}, \quad \hat{r} = \frac{r}{2} - \frac{\sigma^2}{12}$$

It follows that the Asian call option with geometric mean is equivalent to a standard call option in a market with parameters $\hat{r}$, $\hat{\sigma}$. Hence, by the Black-Scholes formula,

$$\Pi_Y(0) = S(0)\Phi(a) - K\,e^{-\hat{r}T}\Phi(b)$$

where

$$a = \frac{\log\frac{S(0)}{K} + (\hat{r} - \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}, \quad b = a + \hat{\sigma}\sqrt{T}$$

and Φ is the standard normal distribution. This concludes the first part of the exercise (4 points). Now let $Q = \left(K - \exp\left(\frac{1}{T}\int_0^T \log S(t)\,dt\right)\right)_+$ be the pay-off of the Asian put with geometric mean. Using the put-call parity for the standard call/put options we have

$$\Pi_Y(0) - \Pi_Q(0) = S(0) - Ke^{-\hat{r}T}$$

Note that the put call parity depends on the volatility of the asset. This concludes the second part of the exercise (1 point)

2 Two assets correlation option

Let $S_1(t), S_2(t)$ be the prices at time t of two stocks. A two asset correlation call option with strikes K_1, K_2 and maturity T is a European style derivative with pay-off

$$Y = \begin{cases} \max\{S_2(T) - K_2, 0\} & \text{if } S_1(T) > K_1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that the risk-free rate is a constant r and that the prices of the stock are given by the SDEs

$$dS_1(t) = rS_1(t) + \sigma_{11}S_1(t)d\widetilde{W}_1(t) + \sigma_{12}S_1(t)d\widetilde{W}_2(t) \tag{1}$$
$$dS_2(t) = rS_2(t) + \sigma_{21}S_2(t)d\widetilde{W}_1(t) + \sigma_{22}S_2(t)d\widetilde{W}_2(t) \tag{2}$$

where $\{\widetilde{W}_1(t)\}_{t\geq 0}, \{\widetilde{W}_2(t)\}_{t\geq 0}$ are independent Brownian motions in the risk neutral probability measure. Derive the PDE and the terminal condition satisfied by the Black-Scholes pricing function of this derivative (max. 2 points). Derive a formula for the Black-Scholes pricing function of the option in the special case when the stock prices are independent (max. 3 points).

Solution. We look for $v(t,x,y)$ such that

$$\Pi_Y(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}[Y|\mathcal{F}_W(t)] = v(t, S_1(t), S_2(t)). \tag{3}$$

Computing $d(e^{-rt}v(t,S_1(t),S_2(t)))$ using Itô's formula, we find that the drift of the process $\{e^{-rt}v(t,S_1(t),S_2(t)\}_{t\geq 0}$ is zero if and only if v satisfies the PDE

$$\partial_t v_g + r(x\partial_x v_g + y\partial_y v_g) + \frac{1}{2}(\sigma_{11}^2+\sigma_{12}^2)x^2\partial_x^2 v_g + \frac{1}{2}(\sigma_{21}^2+\sigma_{22}^2)y^2\partial_y^2 v_g \\ + (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})xy\partial_{xy}v_g = r\, v_g, \quad t\in(0,T), \quad x,y>0. \tag{4}$$

Hence when v satisfies (4), the process $\{e^{-rt}v(t,S_1(t),s_2(t)\}_{t\geq 0}$ is a martingale. Let g be the pay-off function of the derivative, that is

$$g(x,y) = (y-K_2)_+ H(x-K_1), \tag{5}$$

where $(z)_+ = \max(z,0)$, and H is the Heaviside function. Imposing the terminal condition

$$v(T,x,y) = g(x,y) \tag{6}$$

we obtain

$$\widetilde{\mathbb{E}}[e^{-rT}v(T,S_1(t),S_2(t))|\mathcal{F}_W(t)] = e^{-rT}\widetilde{\mathbb{E}}[Y|\mathcal{F}_W(t)] = e^{-rt}(v(t,S_1(t),S_2(t)),$$

hence (3) holds when v satisfies the terminal value problem (4)-(6). This concludes the first part of the exercise (2 points). For the second part, we use that the 2 stocks are independent if and only if $\sigma_{12}=\sigma_{21}=0$ or $\sigma_{11}=\sigma_{22}=0$. Up to the (trivial) transformation $W_1(t)\leftrightarrow W_2(t)$, the two cases are equivalent, hence we may continue assuming $\sigma_{12}=\sigma_{21}=0$. Letting $\sigma_{11}=\sigma_1$, $\sigma_{22}=\sigma_2$, the stock prices satisfy

$$dS_1(t) = rS_1(t) + \sigma_1 S_1(t)d\widetilde{W}_1(t), \tag{7}$$

$$dS_2(t) = rS_2(t) + \sigma_2 S_2(t)d\widetilde{W}_2(t). \tag{8}$$

Thus the volatility matrix is the diagonal matrix $\sigma = \mathrm{diag}(\sigma_1,\sigma_2)$ and so the pricing function of the derivative is

$$v_g(t,x,y) = \frac{e^{-r\tau}}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} g(xe^{(r-\frac{\sigma_1^2}{2})\tau+\sigma_1\sqrt{\tau}\xi}, ye^{(r-\frac{\sigma_2^2}{2})\tau+\sigma_2\sqrt{\tau}\eta})e^{-\frac{1}{2}(\xi^2+\eta^2)}\,d\xi\,d\eta. \tag{9}$$

Computing the integral with the pay-off function (5) we find

$$v_g(t,x,y) = \frac{e^{-r\tau}}{2\pi}\int_{-A_1(x)}^{\infty} e^{-\frac{1}{2}\xi^2}\,d\xi \int_{-A_2(y)}^{\infty}\Big(y\exp\Big((r-\frac{1}{2}\sigma_2^2)\tau + \sigma_2\sqrt{\tau}\eta - \frac{1}{2}\eta^2\Big) - K_2\Big)d\eta$$

where

$$A_i(z) = \frac{\log \frac{K_i}{z} + (r - \frac{\sigma_i^2}{2})}{\sigma_i \sqrt{\tau}}$$

Hence

$$v_g(t, x, y) = \Phi(A_1(x)) C(t, y, K_2, T)$$

where Φ is the standard normal distribution and $C(t, x, K, T)$ is the Black-Scholes pricing function of the call option with strike K and maturity T. The same result is obtained by solving (4) with the method of variables separation (i.e., looking for solutions of the form $v_g(t, x, y) = h(t, x) f(t, y)$).