Mathematics
Chalmers University of Technology
and Göteborgs University

TMA372, MAN660, PARTIAL DIFFERENTIAL EQUATIONS
ASSIGNMENT 1A

1. Consider the two point boundary value problem

$$
(1) \qquad \begin{cases} -(a(x)u_x)_x + b(x)u_x + c(x)u(x) = f(x) \quad \text{on} \quad (0,1), \\ u(1) = 0, \\ \alpha u(0) + \beta u_x(0) = \gamma. \end{cases}
$$

a. Give a variational formulation of this problem in a suitable space. Formulate the corresponding finite element method with piecewise linear approximation. Write out the elements in the matrices and compute them when a, b, c and f are constant functions. Study in particular how the Robin boundary condition is approximated by the finite element method.

b. Prove an *a priori* and an *a posteriori* error estimate under the assumptions that $c \geq 0$ and $b = 0$. Formulate an adaptive algorithm based on the *a posteriori* error estimate.

c. Assume $b = 0$ and $c \geq 0$ and formulate the minimization problem which is equivalent to (1). Show that they are indeed equivalent.

d. Assume both b and c are identically zero, a is constant and $\alpha = 1$, $\beta = \gamma = 0$. The corresponding Green's function $G(x, y)$ associated with a delta function $\delta_y(x) = \delta(x - y)$ at y, is defined by

$$
-(aG_x)_x = \delta_y, \qquad G(0, y) = G(1, y) = 0.
$$

Show that the solution $u(x)$ corresponding to the right hand side $f(x)$ is given by

$$
u(x) = \int_0^1 G(x, y) f(y) dy.
$$

Solve the equation for the Green's function and note that if y is a nodal point then $G(x, y)$ is in the finite element space. Use this to prove that the error at the nodes is in fact zero. This is a surprising one dimensional effect. Hint: See also problems 8.9 and 8.10, in the text book.