TMA372/MMG800 PARTIAL DIFFERENTIAL EQUATIONS ASSIGNMENT 2 AND COMPUTER PROJECT

1. Grading policy of the assignment

For each of the following 5 tasks you obtain 1 point.
You need 1 point to pass this assignment.
Your points beyond 1 will be counted as your bonus points.

Tasks:

- Solve all exercises a)-c) in the theory part (1 point).
- As for selected applications:
  - State a complete boundary value problem, explain its physical meaning and derive the weak formulation (1 point).
  - Describe the discretization procedure: triangulation of the domain, derivation of finial linear system of equations. Write a function (for example in MatLab) that computes the stiffness matrix (1 point).
  - Solve the boundary value problem numerically, visualize the results and explain the plots (1 point).
  - Derive the stability analysis and error estimates (1 point).

2. Theory

Consider the heat equation:

$$\begin{aligned} u_t - \Delta u =& 0, \qquad x \in \Omega, \quad t > 0, \\ u =& 0, \qquad x \in \partial\Omega, \quad t > 0, \\ u(x,0) =& u_0, \qquad x \in \Omega. \end{aligned}$$

a) Show the following stability estimates:

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2\, ds \le \|u_0\|^2, \qquad t > 0,$$

$$\|\Delta u\| \le \frac{1}{t}\|u_0\|, \qquad t > 0.$$

The latter estimate is the so called parabolic smoothing estimate (or strong stability), that describes the fact that the solution is smother than the initial data (it gains regularity).
b) How do these estimates change if you substitute $u_{xx} + 4u_{yy}$ for Δu ?
c) Solve the problem with $\Omega = [0, 1]$ using a Fourier series and study how fast the coefficients for the different Fourier modes decay. Prove the smoothing estimate by using this Fourier series representation of the exact solution.

3. Selected applications

Select one of the following applied problems. The objective is to solve an applied problem of interest using a finite element method implemented in a software of your choice, to evaluate the results obtained and draw some conclusions concerning the nature of the exact solution and the numerical approximation. Use your fantasy and focus on features of interest. Note that the problems are not precisely formulated. You thus have to think of:

- An interesting real world problem.
- Mathematical modeling including for instance the choice of boundary conditions and truncation of the computational domain in case of unbounded domains.
- Computational aspects.
- Analytical aspects, seek to simplify the model so that it is possible to obtain an analytical solution. Solve the simplified problem and think about the extra assumptions you have made, are these realistic?

3.1. **Convection-diffusion-absorption/reaction.** Consider a 2d convection-diffusion-absorption/reaction problem of the form

$$\alpha u + \beta \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) = f,$$

together with suitable boundary conditions on the boundary Γ of Ω, where u is an unknown concentration, $\epsilon = \epsilon(x)$ is a given (small) diffusion coefficient, $\beta = \beta(x)$ is a given velocity field, $\alpha = \alpha(x)$ is a given absorption/reaction coefficient and $f = f(x)$ is a given production term. Solve a convection-dominated problem of this form for instance related to pollution control, where f is a delta-function at some point $P \in \Omega$. Determine for instance the width of the "smoke plume" and compare with theory.

3.2. **Electrostatics.** Consider the basic problem of 2d electrostatics

$$\begin{aligned} \nabla \cdot (\epsilon E) &= \rho, \\ E &= -\nabla \phi, \end{aligned}$$

together with suitable boundary conditions corresponding to a part of the boundary of Ω being a perfect conductor and the remaining part being insulated. Here E is the electric field, ϕ the electric potential, $\epsilon = \epsilon(x)$ the dielectricity coefficient, and ρ the charge density. Solve a problem of this form in a configuration of interest for instance with the boundary containing a sharp non-convex corner. Study the behavior of the electric field in the vicinity of the corner and compare with theory.

3.3. **2d fluid flow.** The velocity $u = (u_1, u_2)$ of an incompressible irrotational 2d fluid may be expressed through a potential ϕ by $u = \nabla \phi$. Coupled with the incompressibility equation $\nabla \cdot u = 0$ this gives the Laplace equation for ϕ:

$$\nabla \cdot (\nabla \phi) = \Delta \phi \quad = \quad 0,$$

together with suitable boundary conditions expressing for instance that $u \cdot n = 0$ on solid boundaries. Note that it is not possible to use Neumann conditions on the entire boundary. Solve a problem of the following type, using a potential:

(a) flow through a 2d nozzle
(b) flow around a disc or wing profile

Use the gradient plot to visualize the flow.

3.4. **Heat conduction.** Consider the 2d stationary heat equation

$$\nabla \cdot q = f, \quad q = -\kappa \nabla u,$$

together with suitable boundary conditions, where u is the temperature, q the heat flow, κ the heat conduction coefficient and f a given production term. Solve for instance a problem of this form modeling a hot water pipe buried in a half space and determine the temperature on the boundary of the half space above the pipe using a Robin boundary condition on the surface.

3.5. **Quantum physics.** Consider the 2d stationary Schrödinger eigenvalue problem

$$-\frac{\hbar^2}{2m}\Delta u + V(x)u = \lambda u,$$

where V is a given potential (choose a nonconstant function), $\hbar$ is Planck's constant divided by 2π and m is the particle mass. Give a quantum physical interpretation of the eigenvalues and corresponding eigenfunctions determined by this equation. Normalize the constants and solve the problem for some suitable domain and potential. Discuss your computational results from a quantum physical viewpoint.