TMA 401/MAN 670 Functional Analysis 2001/2002

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Problem 2

hand in at latest April 26

1. Show that the mapping

$$\|(x,y)\| = (\sqrt{|x|} + \sqrt{|y|})^2$$

does not define a norm in $\mathbf{R}^2$.

2. Prove that any finite-dimensional subspace of a normed space is closed. Show that this is not true for every subspace of a normed space by proving the following: l^1 is a subspace of l^2 which is not closed in l^2.

3. Assume $0 < L < \sqrt{(\sqrt{5}-1)/2}$. Show that there exists a unique $u \in \mathcal{C}[0,1]$ such that

$$u(x) = \int_0^L \sqrt{1+(x-y)^2}\cos(u(y))\,dy + \sin(e^{-x}), \quad 0 \le x \le L.$$

4. Let x and y be elements in a complex vector space with inner product and assume that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Does this imply that $\langle x, y\rangle = 0$? What happens if the complex vector space is replaced by a real vector space?

5. The operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is defined by the equation

$$Tu(x) = u(x) + \int_0^x u(s)\,ds, \quad 0 \le x \le 1.$$

Prove that $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{C}[0,1]$. Finally determine the inverse T^{-1} of T and show that T^{-1} is a bounded operator.