TMA 401/MAN 670 Functional Analysis 2002/2003

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# 1 Exercises

This is a collection of problems that has appeared in the course. Some of them has been given on written examinations during the last five years.

## 1.1 Vector spaces

*Key words:* vector space, linear combination, linear independence, basis, dimension

1. Check if the following sets with the proposed addition $\oplus$ and multiplication by scalar $\odot$ defines vector spaces:

   (a) $E = \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x > 0\}$ and $F = \mathbb{R}$ with
   $$x \oplus y = xy \ \text{ for all } \ x, y \in E$$
   and
   $$\alpha \odot x = x^\alpha \ \text{ for all } \ \alpha \in F,\, x \in E.$$

   (b) $E = \mathbb{C}$ and $F = \mathbb{C}$ with
   $$x \oplus y = x + y \ \text{ for all } \ x, y \in E$$
   and
   $$\alpha x = (\operatorname{Re}\alpha)x \ \text{ for all } \ \alpha \in F,\, x \in E.$$

2. Let $x$ be an element of a vector space and $\lambda$ a scalar. Show that

   (a) $0x = \mathbf{0}$
   (b) $(-1)x = -x$
   (c) $\lambda \neq 0$ and $\lambda x = \mathbf{0}$ implies $x = \mathbf{0}$
   (d) $x \neq 0$ and $\lambda x = \mathbf{0}$ implies $\lambda = 0$

3. Let $E$ be a vector space such that there exist a basis with finitely many vectors. Show that the dimension of $E$ is uniquely defined.

4. Let $x_1, \ldots, x_n$ be a basis for a complex vector space $E$. Find a basis for $E$ as a real vector space.

5. Let $x_1, \ldots, x_n$ be a set of linearly dependent vectors in a complex vector space $E$. Is this set linearly dependent in $E$ if $E$ is regarded as a real vector space?

6. Show that the functions $f_n(x) = e^{nx}$, $n = 1, 2, \ldots$, defined on $\mathbb{R}$ are linearly independent.

7. Show that the functions $f_n(x) = \cos nx$, $n = 1, 2, \ldots$, defined on $[-\pi, \pi]$ are linearly independent.

8. In $C[-1, 1]$ consider the sets $U$ and $V$ consisting of odd and even functions in $C[-1, 1]$ respectively. Show that $U$ and $V$ are subspaces and that $U \bigcap V = \{0\}$. Show that every $f \in C[-1, 1]$ can be written in the form $f = f_1 + f_2$, where $f_1 \in U$ and $f_2 \in V$, and that this decomposition is unique.

9. Let $E = C([0, 1])$. Show that

   (a) if $a_k$, $k = 1, \ldots, n$ are $n$ distinct points in $[0, 1]$ then the functions
   $$x \mapsto |x - a_k|, \quad k = 1, \ldots, n$$
   are linearly independent on $E$,

   (b) the function
   $$(x, y) \mapsto |x - y|$$
   on $[0, 1] \times [0, 1]$ cannot be written as a finite sum
   $$\Sigma_{i=1}^{n} v_i(x) w_i(y),$$
   where $v_i, w_i \in E$, $i = 1, \ldots, n$.

10. Prove that the vector space $C([0, 1])$ has infinite dimension.

11. Prove that the vector space $C^\infty(\mathbb{R})$ has infinite dimension.

12. Prove that the vector spaces $l^p$ are infinite-dimensional for $p \in [1, \infty)$.

13. Let $l^0$ consist of all sequences $(x_n)_{n=1}^{\infty}$, $x_n \in \mathbb{R}$, where at most finitely many $x_n$:s are different from 0. Show that $l^0$ is a vector space with the usual addition and multiplication with scalar operations for sequence spaces. Also give a basis for $l^0$.

14. Let $F$ be a subspace of a vector space $E$. The **coset** of an element $x \in E$ with respect to $F$ is denoted by $x + F$ and is defined to be the set
    $$x + F = \{x + y : y \in F\}.$$
    Show that under the algebraic operations
    $$(x + F) + (y + F) = (x + y) + F$$
    $$\alpha(x + F) = \alpha x + F$$
    these cosets constitute the elements of a vector space. This vector space is called the **quotient space of** $E$ **by** $F$ and is denoted by $E/F$. Its dimension is called the **codimension** of $F$ and is denoted by $\operatorname{codim} F$. Now let $E = \mathbb{R}^3$ and $F = \{(0, 0, z) : z \in \mathbb{R}\}$. Find

(a) $E/F$
(b) $E/E$
(c) $E/\{0\}$

15. Show that $C([c,d])$ is a subspace of $C([a,b])$ (in a natural way) if $[c,d] \subset [a,b]$.

16. Assume $M$ and $N$ are subspaces of a vector space $V$. When is $M \bigcup N$ a subspace?

17. Let $T : E \to F$ be a linear mapping from the vector space $E$ into the vector space $F$. Show that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are vector spaces.

18. Show that linear mappings preserve linear dependence.

19. Let $T$ be a linear bijection between two vector spaces $E$ and $F$. Assume that $E$ is finite-dimensional. Show that also $F$ is finite-dimensional and that $\dim E = \dim F$.

20. The **convex hull** $\hat{S}$ of a set $S$ is defined as the intersection of all convex sets containing $S$.

    (a) Show that $\hat{S}$ is convex.

    (b) If $S \subset R$ and $R$ convex, show that $\hat{S} \subset R$.

    (c) A **convex combination** of elements $x_1, \ldots, x_n$ of a vector space is a linear combination $\Sigma a_i x_i$ with $a_i \geq 0$ for each $i$ and $\Sigma a_i = 1$. If $R$ is a convex set, show that any convex combination of a finite number of elements of $R$ belongs to $R$.

    (d) Show that for any set $S$, $\hat{S}$ equals the set of all convex combinations of finitely many elements of $S$.

## 1.2 Normed spaces

*Key words:* norm, convergence in normed space, equivalence of norms, open/closed ball, open/closed set, closure of set, dense subset, compact set

1. Show that in any normed space
   (a) a convergent sequence has a unique limit;
   (b) if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$;
   (c) if $x_n \to x$ and $\lambda_n \to \lambda$ ($\lambda_n, \lambda$ are scalars) then $\lambda_n x_n \to \lambda x$.

2. Let $E$ be a normed space. Prove that
$$\|x\| \leq \max(\|x-y\|, \|x+y\|),\ x, y \in E.$$
   Give an example of a normed space $E$ and an $x \in E$, such that equality occurs for a suitable $y \neq 0$.

3. Let $x_1, \ldots, x_n$ be linearly independent vectors in a normed space $E$. Show that there exists a $c > 0$ such that
$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq c(|\alpha_1| + \ldots + |\alpha_n|),$$
   for all scalars $\alpha_i,\ 1 \leq i \leq n$. Conclude from this that any two norms on $E$ are equivalent, if $E$ is finite dimensional.

4. Show that equivalent norms define the same opens sets and Cauchy sequences.

5. Show that the norms $\| \ \|_1$ and $\| \ \|_\infty$ are not equivalent in the vector space $C([0,1])$ where
$$\|f\|_1 = \int_0^1 |f(t)|\, dt$$
   and
$$\|f\|_\infty = \max_{t \in [0,1]} |f(t)|$$
   for $f \in C([0,1])$.

6. Given a set $X$. A function $d : X \times X \to [0, \infty)$ is called a **metric** on $X$ if $d$ satisfies the conditions
   (a) $d(x, y) = 0$ iff $x = y$
   (b) $d(x, y) = d(y, x)$ for all $x, y \in X$
   (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

   Show that if $E$ is a vector space with norm $\| \cdot \|$ then
$$d(x, y) = \|x - y\| \quad x, y \in E$$
   defines a metric on $E$.

7. Let $(X, d)$ be a metric space. Show that $d_1$ given by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \text{for} \quad x, y \in X$$

is a metric on $X$. Show that the metrics $d$ and $d_1$ yield the same open sets.

8. Give an example of a metric on a vector space that is not given by a norm.

9. Show that the open balls $B(x, r)$ in a normed space are open sets. Also show that the closed balls are closed sets.

10. A subset $A$ of a vector space $E$ is called **convex** if

$$\alpha x + (1 - \alpha) y \in A \;\; \text{for all} \;\; x, y \in A, \;\; \alpha \in [0, 1].$$

If $E$ is a normed space show that the closed and open unit balls $\bar{B}(0, 1)$ and $B(0, 1)$ are convex.

11. Set $\phi : \mathbb{R}^2 \to [0, \infty)$, where

$$\phi(x, y) = (\sqrt{|x|} + \sqrt{|y|})^2.$$

Show that $\phi$ does not define a norm in $\mathbb{R}^2$.

12. Let $U$ be a bounded open convex and symmetric (i.e. $U = (-1)U$) set in $\mathbb{R}^2$ containing the origin and set

$$\|(x, y)\| = \inf\{\lambda > 0 : (x, y) \in \lambda U\},$$

where $\lambda U = \{(\lambda x, \lambda y) : (x, y) \in U\}$ for $\lambda \in \mathbb{R}$. Show that $\| \; \|$ defines a norm on $\mathbb{R}^2$. Conclude that all norms on $\mathbb{R}^2$ are given in this way.

13. Find a sequence $(x_1, x_2, \dots)$ such that $x_n \to 0$ as $n \to \infty$ but is not in any $l^p$, where $1 \le p < \infty$. Find a sequence $(x_1, x_2, \dots)$ which is in $l^p$ with $p > 1$ but not in $l^1$. Is $l^p \setminus l^q = \emptyset$ if $p > q$?

14. Give an example of a subspace in $l^2$ that is not closed.

15. Let $1 \le r < p < 2r$ and assume that the sequence $(x_1, x_2, \dots)$ satisfies

$$\Sigma_{n=1}^{\infty} n |x_n|^p < \infty.$$

Show that $(x_1, x_2, \dots) \in l^r$.

16. Show that

$$\lim_{j \to \infty} \Sigma_{n=1}^{\infty} \frac{x_n}{j + n} = 0$$

for all $(x_1, x_2, \dots) \in l^2$.

17. Let $f(x) = \sin x$ for $0 \le x \le 1$. Find a sequence of polynomials $p_n(x)$, $0 \le x \le 1$, $n \in \mathbb{N}$ of degree $n$, which converges to $f$ in $C([0, 1])$.

18. Show that every continuous function $f$ on $[0,1]$ can be uniformly approximated by polynomials, i.e. for each $\epsilon > 0$ there is a polynomial $p$ such that $\max_{t\in[0,1]} |f(t) - p(t)| < \epsilon$. This statement is known as the **Weierstrass approximation theorem**[1].

19. Show that if $A$ is dense in $B$ and $B$ is dense in $C$ then $A$ is dense in $C$.

20. Prove or disprove: if $A$ is dense in $B$ then for any set $C$, $A \bigcap C$ is dense in $B \bigcap C$.

21. Let $E$ be a normed space. $E$ is called **separable** if there exists a countable dense subset in $E$. Show that
    (a) $\mathbb{R}$ is separable
    (b) $l^p$ is separable for $p \in [1, \infty)$
    (c) $l^\infty$ is not separable[2]
    (d) $C([0,1])$ is separable

22. Let $E$ be a normed space and $(x_n)_{n=1}^\infty$ a countable dense subset in $E$. Given $\epsilon > 0$ show that
$$E \setminus \{0\} \subset \bigcup_{n=1}^{\infty} B(x_n, \epsilon\|x_n\|).$$

23. Show that every finite set is compact.

24. Show that $\mathbb{R}^n$ and $\bar{B}(0,1) \bigcap \{(x_1, \dots, x_n) : x_1 < 1/2\}$ are not compact sets using the definition of compactness.

25. Construct a set in $\mathbb{R}^2$ which has finite area but is not relatively compact. Generalize to $\mathbb{R}^n$.

26. Prove that any finite-dimensional subspace of a normed linear space is closed.

27. If $S$ is a relatively compact set, prove that its convex hull is relatively compact.

28. Let $F$ be a subspace of a normed space $E$ and suppose $x_0 \in E \setminus F$. Furthermore suppose $x_0$ possesses a nearest point in $F$ (i.e. there is a $y_0 \in F$ such that $\|y - x_0\| \geq \|y_0 - x_0\|$ for all $y \in F$).
    (a) Prove that there is an $x_1 \in E$ such that $\|x_1\| = 1$ and $\|y - x_1\| \geq 1$ for all $y \in F$.
    (b) In addition, suppose $\mathrm{Span}(\{x_0\} \bigcup F) = E$. Show that every $x \in E$ possesses a nearest point in $F$.

[1] Hint: One way to prove the claim is to use the so called Bernstein polynomials, more precisely set
$$B_n f(x) = \Sigma_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k} f(\frac{k}{n}),\ x \in [0,1],\ n = 1, 2, \dots$$
Show that $B_n f \to f$ in $C([0,1])$ as $n \to \infty$.

[2] Assume that it is separable and construct a function that has $l^\infty$-distance $\geq 1$ to each function in the supposed countable dense set.

29. (**Riesz lemma**) Suppose $E$ is a normed space and let $F$ be a proper closed subspace of $E$. Furthermore let $\epsilon$ be a given positive real number. Show that there is a vector $x_1 \in E$ such that $\|x_1\| = 1$ and $\|y - x_1\| > 1 - \epsilon$ for every $y \in F$.

30. Let $E$ be a normed space. Show that the unit sphere $\{x \in E; \|x\| = 1\}$ is compact if and only if $E$ is of finite dimension.

31. Let $F$ be a closed subspace of a normed space $E$, where $\|\cdot\|$ denotes the norm. Show that $\|\cdot\|_0$ defines a norm on the quotient space $E/F$ if

$$\|\tilde{x}\|_0 = \inf_{x \in \tilde{x}} \|x\|.$$

## 1.3 Banach spaces

*Key words:* Cauchy sequence, complete space, Banach space, convergent/absolutely convergent series, linear mapping, null space of a linear mapping, range and graph of a mapping, continuous mapping, bounded linear mapping, completion of a normed space, $L^p$-spaces

1. Prove that convergence in $L^2([0,1])$ implies convergence in $L^1([0,1])$.

2. For any $n \in \mathbb{Z}_+$ set
$$f_n(x) = \begin{cases} \sqrt{n} & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1. \end{cases}$$
Prove that $f_n \to 0$ in $L^1([0,1])$ but not in $L^2([0,1])$.

3. Let $f \in L^1(\mathbb{R})$. Can we conclude that $f(x) \to 0$ for $|x| \to \infty$? Can we find $a, b \in \mathbb{R}$ such that $|f(x)| \le b$ for $|x| \ge a$?

4. Which of the following sequences of real functions ($n \in \mathbb{N}$)

   (a) $f_n = \frac{1}{n}\chi_{(0,n)}$

   (b) $f_n = \chi_{(n,n+1)}$

   (c) $f_n = n\chi_{[0,\frac{1}{n}]}$

   (d) $f_n = \chi_{[j2^{-k},(j+1)2^{-k}]}$ where $0 \le j < 2^k$ and $n = j + 2^k$

   converges to 0

   (a) uniformly on $\mathbb{R}$

   (b) point-wise on $\mathbb{R}$

   (c) almost everywhere on $\mathbb{R}$

   (d) in $L^1(\mathbb{R})$.

   Which of these modes of convergence implies which others?

5. Let $f \in L^p(\mathbb{R})$ for $p \in [1,\infty)$ and $\lambda > 0$. Prove the inequality
$$|\{x \in \mathbb{R} : |f(x)| > \lambda\}| \le (\frac{\|f\|_p}{\lambda})^p,$$
where $|A|$ denotes the (Lebesgue) measure of the set $A \subset \mathbb{R}$.

6. Let $f \in C[0,1]$. Show that
$$\|f\|_p \to \|f\|_\infty \text{ for } p \to \infty.$$

7. Consider the set of all rational numbers $p/q \in (0,1)$ with denominator $q \leq n$; call them $r_{n1}, r_{n2}, \ldots, r_{nK}$ (where $K$ depends on $n$). Define a function $g_n$ by

$$g_n(x) = \Sigma_{i=1}^{K} \phi_n(x - r_{ni}),$$

where $\phi_n(u) = 1 - e^n u$ for $|u| \leq e^{-n}$, $\phi_n(u) = 0$ for $|u| > e^{-n}$. Sketch the graph of $g_n$. Show that $g_n \in C([0,1])$, $\int_0^1 |g_n|^2 \, dx \to 0$ as $n \to \infty$, and and $g_n(x) \to \chi_{\mathbb{Q}}(x)$ for rational $x$.

8. Let $t > 0$ and define the operator

$$P_t : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$$

by the equation

$$(P_t f)(x) = \int_{\mathbb{R}^n} e^{-|x-y|/2t} f(y) dy.$$

Prove that $P_s P_t = P_{s+t}$

9. Show that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and has a convergent subsequence then $(x_n)_{n=1}^{\infty}$ is convergent.

10. Assume that $(x_n)_{n=1}^{\infty}$ is a sequence in a Banach space such that for any $\epsilon > 0$ there is a convergent sequence $(y_n)_{n=1}^{\infty}$ such that $\|y_n - x_n\| < \epsilon$ for all $n$. Prove that $(x_n)_{n=1}^{\infty}$ is convergent. Give an example to show that the statement becomes false if Banach space is replaced by normed space.

11. Let $l_c^{\infty}$ denote the vector space with all convergent sequences $(x_n)_{n=1}^{\infty}$ of complex numbers equipped with the norm

$$\|(x_n)_{n=1}^{\infty}\|_{l_c^{\infty}} = \sup_n |x_n|.$$

Show that the space $l_c^{\infty}$ is complete.

12. Define $C_2^1([0,1])$ to be the space of continuously differentiable functions on $[0,1]$, with norm $\|f\| = (\int_0^1 (|f|^2 + |f'|^2) \, dx)^{1/2}$. Show that this is a proper definition of norm. Is this normed space complete?

13. What conditions must the function $r$ satisfy in order that

$$\|f\| = \sup\{|f(x) r(x)| : 0 \leq x \leq 1\}$$

should define a norm on the vector space $C([0,1])$?

14. Let $BC([0,\infty))$ be the set of functions continuous for $x \geq 0$ and bounded. Show that for each $a > 0$, $\|f\|_a = (\int_0^{\infty} e^{-ax} |f(x)|^2 \, dx)^{1/2}$ defines a norm on $BC([0,\infty))$, and $\|\cdot\|_a$ is not equivalent to $\|\cdot\|_b$ if $0 < b < a$. What about the case $a = 0$?

15. Show that every finite-dimensional normed space is complete.

16. Set $f_k(x) = \dfrac{\sin kx}{k^2}$, $0 \le x \le 1$, $k \in \mathbb{Z}_+$. Prove that the series $\Sigma_{k=1}^{\infty} f_k$ converges in $C([0,1])$.

17. Set for any $n \in \mathbb{Z}_+$, $f_n(x) = x^n - x^{n+1}$ and $g_n(x) = x^n - x^{2n}$ if $0 \le x \le 1$. Is any of the sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ convergent in $C([0,1])$?

18. Let $M = \{x \in C([0,1]) : x(2^{-n}) = 0 \;\text{ all }\; n \in \mathbb{Z}_+\}$. Prove that $M$ is a closed subset of $C([0,1])$.

19. Let $M = \{(x_n)_{n=1}^{\infty} \in c_0 : \Sigma_{n=1}^{\infty} 2^{-n} x_n = 0\} \subset c_0 \equiv \{(x_n)_{n=1}^{\infty} \in l^{\infty} : \lim_{n\to\infty} x_n = 0\}$. Show that $M$ is a closed subspace in $c_0$.

20. Let $E$ denote a normed space of finite dimension and let $e_1, \dots, e_n$ be a basis of $E$. Set
$$f(x) = \Sigma_{k=1}^{n} x_k e_k,\; x = (x_1, \dots, x_k) \in \mathbb{R}^n.$$
Show that $f$ is continuous. Conclude from this that any two norms on $E$ are equivalent.

21. Let $E$ be a normed space and assume that $E \neq \{0\}$. Prove that there do not exist bounded linear operators $A$ and $B$ on $E$ such that $AB - BA = I$.

22. Set $(Ax)(t) = x'(t)$ and $(Bx)(t) = tx(t)$, $0 < t < 1$, for $x \in C^{\infty}(]0,1[)$. Prove that $AB - BA = I$. Is it possible to find a norm on $C^{\infty}(]0,1[)$ such that $A$ and $B$ are bounded operators with respect to this norm[3]?

23. Let $E$ and $F$ be normed spaces and $T : E \to F$ a continuous mapping. Show that the $T(A)$ is compact in $F$ if $A$ is a compact set in $E$.

24. Let $T : E \to \mathbb{R}$ be a continuous mapping from a normed space $E$. Moreover let $A$ be a compact set in $E$. Show that $T$ attains its maximum and minimum on $A$.

25. Let $A : X \to X$ be a continuous mapping and assume $Ax \neq 0$ for all $x \in X$. Show that the mapping $B : x \mapsto Ax/\|Ax\|$ is continuous on $X$.

26. Find the norm of the linear functional
$$(x, y) \mapsto x - 7y$$
on $\mathbb{R}^2$ with respect to the norms $l^p$ for $p = 1, 2$ and $\infty$.

27. For what values of the constant $a$ does
$$u \mapsto \int_0^1 x^a u(x)\, dx$$
define a mapping $C([0,1]) \to \mathbb{C}$? For what values of $a$ does it define a mapping $L^2([0,1]) \to \mathbb{C}$?

[3]Hint: Show that $A^n B - BA^n = nA^{n-1}$ for $n = 1, 2, \dots$

28. Show that the equation

$$\begin{cases} (Af)(x) = \int_{-\infty}^{+\infty} f(y)e^{-|x-y|}\, dy, & x \in \mathbb{R} \\ f \in L^2(\mathbb{R}) \end{cases}$$

defines a bounded linear operator $A$ on $L^2(\mathbb{R})$.

29. Prove that any linear mapping from a finite-dimensional vector space into an arbitrary vector space must be continuous.

30. Let $E$ be a normed space and $L$ a linear functional on $E$. Furthermore, suppose there is a unit vector $x_0 \in E$ such that $\|x_0 - y\| \geq 1$ for every $y \in \mathcal{N}(L)$. Prove that $|Lx_0| = \|L\|$.

31. Find all linear mappings of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $n, m \in \mathbb{Z}_+$.

32. Let $A, B$ be two linear operators defined on a vector space $E$. Show that $E$ must be infinite-dimensional if

$$AB = I \neq BA,$$

where $I$ denotes the identity mapping on $E$. Give an example of such operators $A$ and $B$ on a vector space $E$.

33. Let $E$ be a vector space and $f : E \to \mathbb{R}$ a linear mapping. Suppose $x_0 \in E$ and $f(x_0) \neq 0$. Prove that any $x \in E$ may be written as $x = y + \alpha x_0$, where $\alpha$ is a scalar and $y \in \mathcal{N}(f)$. Show that this representation is unique.

34. Let $f$ and $g$ be two functionals on a vector space such that $\mathcal{N}(g) \subset \mathcal{N}(f)$. Prove that $f = \alpha g$, where $\alpha$ is a scalar.

35. Show that for any linear operator $A$ on a $n$-dimensional vector space $E$, there are scalars $\alpha_0, \ldots, \alpha_{n^2}$, not all of them zero, such that

$$\Sigma_{k=0}^{n^2} \alpha_k A^k$$

is the zero operator.

36. Let $B$ and $C$ be linear operators on a finite-dimensional vector space $E$ and suppose $\mathcal{N}(B) \subset \mathcal{N}(C)$. Show that there is a linear operator $A$ on $E$ such that $C = AB$.

37. Let $E$ be a vector space of finite dimension and suppose $A : E \to E$ is a linear operator. Prove that $\mathcal{N}(A) = \{0\}$ if and only if $\mathcal{R}(A) = E$. Show that this is not true for vector spaces of infinite dimension.

38. Let $E$ be a real normed space and let $T : E \to \mathbb{R}$ be a linear functional. Assume that $\mathcal{N}(T) \neq E$. Show that for all $x \in E$

$$\min_{y \in \mathcal{N}(T)} \|x - y\| = \frac{|Tx|}{\|T\|}.$$

39. Show that the operator $T$ on $C([0,1])$, where

$$(Tf)(t) = tf(t), \quad t \in [0,1],$$

is a bounded linear operator on $C([0,1])$.

40. Let $A_n, A, B_n, B$ be bounded linear operators on a Banach space $X$. Show that $A_n \to A$ and $B_n \to B$ in $\mathcal{B}(X,X)$ implies $A_nB_n \to AB$ in $\mathcal{B}(X,X)$.

41. Let $A : X \to X$ be a bounded linear operator on a Banach space $X$. Show that $\Sigma_{n=0}^{\infty} \frac{1}{n!} A^n$ converges in $\mathcal{B}(X,X)$. Denote its sum by $e^A$. Show that for any integer $n > 0$, $(e^A)^n = e^{nA}$. Show that $e^O = I$ where $O$ is the zero operator. Show that $e^A$ is always invertible (even if $A$ is not) and its inverse operator is $e^{-A}$. Show that if $AB = BA$, then $e^{A+B} = e^A e^B$. Show that $e^{A+B} = e^A e^B$ is not true in general.

42. Let $A, B$ be invertible bounded linear operators on a Banach space $X$ with $\|B^{-1}\|\|A - B\| < 1$. Show that if

$$\begin{cases} Ax = b \\ By = b \end{cases}$$

then

$$\|x - y\| \leq \frac{\|B^{-1}\|\|A - B\|}{1 - \|B^{-1}\|\|A - B\|}\|y\|.$$

Moreover also show that

$$\|x - y\| \leq \frac{\|B^{-1}\|^2\|A - B\|}{1 - \|B^{-1}\|\|A - B\|}\|b\|.$$

43. Let $T$ be a bounded linear operator from a normed space $E$ onto a normed space $F$. Assume that there is a constant $C > 0$ such that

$$\|Tx\| \geq C\|x\|$$

for all $x \in E$. Show that the inverse operator $T^{-1} : F \to E$ exists as a mapping and is a bounded linear operator.

44. Let $T : C([0,1]) \to C([0,1])$ be defined by

$$(Tf)(t) = \int_0^t f(s)\, ds.$$

Find $\mathcal{R}(T)$ and $T^{-1} : \mathcal{R}(T) \to C([0,1])$ satisfying $T^{-1}T = I_{C([0,1])}$. Is $T^{-1}$ linear and bounded?

45. The operator $A : C([0,1]) \to C([0,1])$ is defined by the equation

$$(Af)(t) = f(t) + \int_0^t f(s)\, ds \quad 0 \leq t \leq 1.$$

Prove that $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = C([0,1])$. Finally determine the inverse $A^{-1}$ of $A$ and show that $A^{-1}$ is a bounded operator.

46. Let $A$ be an $r \times n$-matrix with real entries. Consider $A$ as a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^r$. Calculate or give an upper bound for the operator norm of $A$ in

    (a) $\mathcal{B}(l^1, l^1)$

    (b) $\mathcal{B}(l^\infty, l^\infty)$

47. Let $F$ be a subspace of a vector space $E$ and let $f$ be a functional on $E$ such that $f(F)$ is not the whole scalar field of $E$. Show that $f(x) = 0$ for all $x \in F$.

48. Let $k \in \mathbb{Z}_+$ and set $L_k(f) = \int_0^\pi f(t) \sin kt \, dt$ for all $f \in C([0, \pi])$. Prove that $\|L_k\| = 2$ for all $k$.

49. Let

$$Lf = \int_0^{1/2} f(t)\, dt - \int_{1/2}^1 f(t)\, dt, \; f \in C([0, 1]).$$

    Prove that $\|L\| = 1$. Prove that there does not exist any $f \in C([0, 1])$ such that $\|f\| = 1$ and $|Lf| = 1$.

50. Let $E$ be a normed space and $A : E \to \mathbb{C}$ a bounded linear functional. Suppose there exists a vector $x_0 \in E$ such that $\|x_0\| = 1$ and $\|x_0 - x\| \geq 1$ for all $x \in \mathcal{N}(A)$. Show that $|Ax_0| = \|A\|$. Moreover, let $F = \{x \in C([0, 1]) : \int_0^{1/2} x(t)\, dt = \int_{1/2}^1 x(t)\, dt\}$. Show that if $x_0 \in C([0, 1])$ and $\|x - x_0\| \geq 1$ for all $x \in F$ then $\|x_0\| > 1$.

51. Show that

$$L(f) = \int_0^1 \frac{1}{\sqrt{x}} f(x)\, dx$$

    defines a bounded linear functional on $C([0, 1])$.

52. Show that

$$L(f) = \int_0^1 \frac{1}{\sqrt[3]{x}} f(x)\, dx$$

    defines a bounded linear functional on $L^2([0, 1])$.

53. (Non-orthogonal projections) A bounded linear operator $P$ on a Banach space $X$ will be called a **projector**[4] if $P^2 = P$.

    (a) Show that $I - P$ is a projector if $P$ is. Show that if $x \in \mathcal{R}(P)$ then $Px = x$, and if $x \in \mathcal{R}(I - P)$ then $Px = 0$.

    (b) Show that for any projector $P$ on a Banach space $X$, the range $\mathcal{R}(P)$ of $P$ is a closed subspace, and is therefore itself a Banach space.

    (c) Show that any $x \in X$ can be uniquely expressed in the form $x = u + v$ with $u \in \mathcal{R}(P)$ and $v \in \mathcal{R}(I - P)$.

[4]Compare projections that are self-adjoint and satisfies $P^2 = P$. By projection we mean orthogonal projection.

54. Let $T$ be a linear mapping from a normed space $V$ into a normed space $W$. Show that the range $\mathcal{R}(T)$ is a subspace of $W$. Show that the null-space (or kernel) $\mathcal{N}(T)$ is a subspace of $V$. If $T$ is bounded, is it true that $T(V)$ and/or $\mathcal{N}(T)$ is closed?

55. Show that if $(x_1^{(n)}, x_2^{(n)}, \dots) \to (x_1, x_2, \dots)$ in $l^p$, then $x_k^{(n)} \to x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$. If $x_k^{(n)} \to x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$, is it true that $(x_1^{(n)}, x_2^{(n)}, \dots) \to (x_1, x_2, \dots)$ in $l^p$?

56. Let $T$ be the linear mapping from $C^\infty(\mathbb{R})$ into itself given by $Tf = f'$. Show that $T$ is surjective. Is $T$ injective?

57. Consider the mapping $T$ from $C[0,1]$ into itself, given by

$$Tf(t) = \int_0^t f(s)\, ds.$$

We assume that $C[0,1]$ is equipped with the sup-norm. Show that $T$ is bounded and find $\|T\|$. Show that $T$ is injective and find $T^{-1} : T(C[0,1]) \to C[0,1]$. Is $T^{-1}$ bounded?

58. Let $T$ be a linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfying that $f \geq 0$ implies that $Tf \geq 0$. Show that

$$\|T(|f|)\| \geq \|Tf\|$$

for all $f \in L^2(\mathbb{R})$. Show that $T$ is bounded.

59. Define, for $h \in \mathbb{R}$, the operator $\tau_h$ on $L^2(\mathbb{R})$ by

$$\tau_h f(x) = f(x-h).$$

Show that $\tau_h$ is bounded.

60. Let $V$ be a Banach space and let $T \in \mathcal{B}(V,V)$ such that $T^{-1}$ exists and belongs to $\mathcal{B}(V,V)$. Show that if $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$, then

$$\|T\| = \|T^{-1}\| = 1,$$

and $\|Tf\| = \|f\|$ for all $f \in V$.

61. Consider the operator

$$Af(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt, \quad x \in [0,1]$$

whenever this expression makes sense. Show that $Af \in L^\infty[0,1]$ if $f \in L^p[0,1]$, $p > 2$. Find the operator $B = A^2$, i.e. find the kernel $k(x,t)$ such that

$$Bf(x) = \int_0^x k(x,t) f(t)\, dt$$

for $f \in L^p[0,1]$, $p > 2$. Show that $B : L^p[0,1] \to L^\infty[0,1]$, $1 \le p \le \infty$ is bounded. Solve the equation

$$(I - A)f(x) = 1$$

formally by a Neumann series, and express $f$ as

$$f(x) = g(x) + Ah(x)$$

where $g$ and $h$ are known functions. Insert and show that this formal solution is a solution.

## 1.4 Fixed point techniques

*Key words:* contractions, Banach's fixed point theorem, Brouwer's fixed point theorem, Schauder's fixed point theorem

1. Show that the Banach fixed point theorem is valid for metric spaces $(X, d)$ as follows: Let $(X, d)$ be a complete[5] metric space and let $F$ be a closed set in $X$. Assume that $T : F \to F$ is a contraction mapping on $F$. Then $T$ has a unique fixed point.

2. Consider the metric space $(X, d)$, where $X = [1, \infty)$ and $d$ the usual distance. Let $T : X \to X$ be given by
$$T(x) = \frac{x}{2} + \frac{1}{x}.$$
Show that $T$ is a contraction and find the minimal contraction constant. Find also the fixed point.

3. Let $T$ be a mapping from a metric space $(X, d)$ into itself such that
$$d(T(x), T(y)) < d(x, y)$$
for all $x, y \in X$, $x \neq y$. Show that $T$ has at most one fixed point. Show[6] that $T$ not necessarily have a fixed point.

4. A mapping $T : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz-condition with constant $k$ if
$$|T(x) - T(y)| \leq k|x - y|$$
for all $x, y \in \mathbb{R}$.
   (a) Is $T$ a contraction?
   (b) If $T$ is a $C^1$–function with bounded derivative, show that $T$ satisfies a Lipschitz-condition.
   (c) If $T$ satisfies a Lipschitz-condition, is $T$ then a $C^1$-function with bounded derivative?
   (d) Assume that $|T(x) - T(y)| \leq k|x - y|^\alpha$ for some $\alpha > 1$. Show that $T$ is a constant.

5. We consider the vector space $\mathbb{R}^n$ with $l^1$-norm and a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $Tx = Cx + b$, where $C = (c_{ij})$ is an $n \times n$-matrix and $b \in \mathbb{R}^n$. Show that $T$ is a contraction if
$$\Sigma_{i=1}^n |c_{ij}| < 1 \text{ for all } j = 1, 2, \ldots, n.$$
If we instead use the $l^2$-norm, show that $T$ is a contraction if
$$\Sigma_{i=1}^n \Sigma_{j=1}^n |c_{ij}|^2 < 1.$$

[5] see footnote to Baire's theorem below
[6] Hint: e.g. consider $T(x) = x + \frac{1}{x}$ for $x \in [1, \infty)$.

6. Use Banach fixed point theorem to find a root (given to four decimal places) of the equation
$$x^2 - \sin^2 x - 1 = 0$$
in the interval $[1, \sqrt{2}]$.

7. Suppose $0 < L < \sqrt{(\sqrt{5}-1)/2}$. Show that there exists a unique $u \in C([0,1])$ such that
$$u(x) = \int_0^L \sqrt{1+(x-y)^2} \cos(u(y))\, dy + \sin(e^{-x}), \quad 0 \leq x \leq L.$$

8. Show that the equation
$$u(x) = \int_0^p \sqrt{1+(x-y)^2} \cos u(y)\, dy + \sin(\pi e^{-4x^2})$$
has a unique solution in $C([0,p])$ for $p > 0$ small enough. Give an upper estimate on $p$?

9. Suppose $\lambda \in \mathbb{C}$. Solve the equation
$$\begin{cases} u(x) - \lambda \int_0^1 xyu(y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0,1]) \end{cases}$$
where $f \in C([0,1])$ is a given function.

10. Suppose $\lambda \in \mathbb{C}$. Solve the equation
$$\begin{cases} u(x) - \lambda \int_0^x xyu(y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0,1]) \end{cases}$$
where $f \in C([0,1])$ is a given function.

11. Suppose $f \in C([0,1])$. Prove that the following equation possesses a unique solution where
$$\begin{cases} u(x) - 5 \int_0^{1-x} u(y) \min(x,y)\, dy = f(x) & 0 \leq x \leq 1 \\ u \in C([0,1]). \end{cases}$$

12. Let $P$ be the set of all ordered pairs $f = (f_1, f_2)$ of real-valued continuous functions on $[0,1]$. Show that $P$ is a Banach space if we define addition and scalar multiplication in the obvious way, and define $\|f\|_P = \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$. Show that the coupled integral equations
$$\begin{cases} u(x) = \lambda \int_0^1 e^{xy} \frac{u(y)}{1+u^2(y)+v^2(y)}\, dy \\ \\ v(x) = \mu \int_0^1 e^{xy} \frac{u(y)v(y)}{1+u^2(y)+v^2(y)}\, dy \end{cases}$$
have no nontrivial solutions if $|\lambda| < 1/2e$ and $|\mu| < 1/e$.

13. Consider the equation[7]

$$3u(x) = x + (u(x))^2 + \int_0^1 |x - u(y)|^{1/2}\, dy.$$

Show that it has a continuous solution $u$ satisfying $0 \leq u(x) \leq 1$ for $0 \leq x \leq 1$.

14. Let $S$ be the set $\{f \in C([0,1]) : \|f\|_\infty \leq 1,\ f(0) = 0,\ f(1) = 1\}$ and the operator $T : S \to S$ defined by $(Tf)(x) = f(x^2)$. Show that $S$ is a closed bounded convex set and that $T$ is a continuous operator with no fixed point.

15. Let $c_0$ denote the vector space

$$c_0 = \{(x_n)_{n=1}^\infty \in l^\infty : \lim_{n\to\infty} x_n = 0\}$$

with the norm

$$\|(x_n)_{n=1}^\infty\|_{c_0} = \max_n |x_n|.$$

Define $T : c_0 \to c_0$ by $T((x_n)_{n=1}^\infty) = (z_n)_{n=1}^\infty$, where

$$\begin{cases} z_1 = \frac{1}{2}(1 + \|(x_n)_{n=1}^\infty\|) \\ z_n = (1 - 2^{-n})x_{n-1},\ n \geq 2 \end{cases}$$

Show that $T$ maps the closed unit ball in $c_0$ into itself and that

$$\|T(x) - T(y)\| < \|x - y\|$$

for all $x, y$, $x \neq y$, in the unit ball in $c_0$. Moreover, show that $T$ have no fixed points in the unit ball in $c_0$.

16. Let $T$ denote the mapping $(x, y) \mapsto (x + y, y - (x + y)^3)$ on $\mathbb{R}^2$. Show that $T$ is an odd mapping, i.e. $T(-x, -y) = -T(x, y)$, and that $(0, 0)$ is the only fixed point of $T$. Moreover show that $(2, -4)$ and $(-2, 4)$ are fixed points of $T^2$. Can $T$ be a contraction?

17. $T$ denote the mapping $(x, y) \mapsto (y^{1/3}, x^{1/3})$ on $\mathbb{R}^2$. What are the fixed points of $T$? What happens when you iterate, starting from various places in $\mathbb{R}^2$ (find out by numerical experiments)? In what regions is $T$ a contraction?

18. Let $T$ be a contraction on a Banach space $E$, i.e.

$$\|Tx - Ty\| \leq \alpha\|x - y\|$$

for all $x, y \in E$ for some $\alpha \in (0, 1)$, and assume that $S$ is a mapping on $E$ such that $\|Tx - Sx\| \leq \lambda$ for all $x \in E$ for some constant $\lambda > 0$. Show that

$$\|T^n x - S^n x\| \leq \lambda \frac{1 - \alpha^n}{1 - \alpha}$$

[7] Hint: Krasnoselskii's fixed point theorem

for $n \in \mathbb{Z}_+$. Show that if $S$ has a fixed point $y$ then

$$\|x - y\| \leq \lambda \frac{1}{1-\alpha},$$

where $x$ is the unique fixed point for $T$. Finally show that if $y_n = S^n y_0$ then

$$\|x - y_n\| \leq \frac{1}{1-\alpha}(\lambda + \alpha^n \|y_0 - Sy_0\|),$$

provided $x$ is the fixed point for $T$. What is the significance of this formula in applications?

19. Consider the equation

$$x(t) - \mu \int_0^1 k(t,s)x(s)\,ds = v(t), \quad t \in [0,1], \tag{1}$$

where $k \in C([0,1] \times [0,1])$ and $v \in C([0,1])$. Moreover assume that

$$\max_{(t,s)\in[0,1]\times[0,1]} |k(t,s)| \leq c.$$

Show that (1) has a unique solution $x \in C([0,1])$ provided $|\mu|c < 1$ using the iterative sequence

$$x_{n+1}(t) = v(t) + \mu \int_0^1 k(t,s)x_n(s)\,ds. \tag{2}$$

Next set

$$Sx(t) = \int_0^1 k(t,s)x(s)\,ds$$

and

$$z_{n+1} = \mu S z_n.$$

Choosing $x_0 = v$ show that (2) yields the so called **Neumann series**

$$x = \lim_{n\to\infty} x_n = v + \mu Sv + \mu^2 S^2 v + \mu^3 S^3 v + \ldots$$

Show that in the Neumann series we can write

$$S^n v(t) = \int_0^1 k_{(n)}(t,s)v(s)\,ds, \quad n = 1,2,3,\ldots$$

where the so called iterated kernel $k_{(n)}$ is given by

$$k_{(n)}(t,s) = \int_0^1 \cdots \int_0^1 k(t,t_1)k(t_1,t_2)\cdots k(t_{n-1},s)\,dt_1 \cdots dt_{n-1}.$$

Show that the solution of (1) can be written

$$x(t) = v(t) + \mu \int_0^1 \tilde{k}(t,s,\mu)v(s)\,ds$$

where

$$\tilde{k}(t,s,\mu) = \Sigma_{j=0}^{\infty} \mu^j k_{(j+1)}(t,s).$$

20. Use the methods in the above problem to solve

$$x(t) - \mu \int_0^1 cx(s)\, ds = v(t), \quad t \in [0,1]$$

where $c$ is a constant.

21. (a) A nonlinear version of the Volterra operator is defined as follows: $(Lu)(x) = \int_0^x K(x,y)f(y,u(y))\, dy$ where $K$ and $f$ are continuous functions, and $|f(y,u) - f(x,v)| \leq N|u-v|$ for all $u, v, x, y$ where $N$ is a constant. Then $L$ maps $C([0,T])$ into itself for any $T > 0$. Give an example to show that $L$ is not a contraction on $C([0,T])$ with the usual norm $\|u\| = \sup |u(x)|$.

    (b) Show that for any $a > 0$, $\|u\|_a = \sup\{e^{-ax}|u(x)| : 0 \leq x \leq T\}$ defines a norm on $C([0,T])$ which is equivalent to the usual norm. Deduce that $C([0,T])$ with the norm $\|\cdot\|_a$ is a Banach space.

    (c) Set $M = \max\{|K(x,y)| : 0 \leq x, y \leq T\}$. Show that $\|Lu - Lv\|_a \leq MN/a(1 - e^{-aT})\|u - v\|_a$ for all $u, v \in C([0,T])$. Deduce that for any $T > 0$ the integral equation $u = Lu + g$, where $g$ is a given continuous function, has a unique solution.

22. (Newton's iteration) Let $f$ be a real $C^2$-function on an interval $[a,b]$, and let $\xi \in (a,b)$ be a simple zero of $f$. Show that Newton's method

$$x_{n+1} = T(x_n) \equiv x_n - \frac{f(x_n)}{f'(x_n)}$$

is a contraction in some neighborhood of $\xi$.

23. (Halley's iteration) In 1694 Edmund Halley, well-known for first computing the orbit of the Halley comet, presented the following algorithm for computing roots of a polynomial. Show that if $f$ is a real $C^3$-function on an interval $[a,b]$, and if $\xi \in (a,b)$ is a simple zero of $f$ then the algorithm

$$x_{n+1} = T(x_n) \equiv x_n - \frac{f(x_n)}{f'(x_n) - \frac{f''(x_n)f(x_n)}{f'(x_n)}}$$

is a contraction in some neighborhood of $\xi$.

24. For each of the following sets give an example of a continuous mapping of the set into itself that has no fixed points:

    (a) the real line $\mathbb{R}$

    (b) the interval $(0,1]$

    (c) the set $[0,1] \bigcup [2,3]$

25. Give an example of a mapping of the closed interval $[0,1]$ into itself that has no fixed points (and hence is not continuous).

26. Let $f : S^1 \to \mathbb{R}$ be a continuous function, where $S^1$ denotes the unit circle centered at the origin. Show that there is an $x \in S^1$ such that $f(x) = f(-x)$. This result is called the **Borsuk-Ulam theorem** for the circle.

27. Let $A$ and $B$ be two bounded plane figures. Show that there is a line dividing each into two parts of equal area.

28. Let $K$ be a closed disc in the plane $\mathbb{R}^2$ and let $C$ be its boundary circle. Assume that the function $f$ is a continuous mapping $K \to \mathbb{R}^2$ such that $f|_C = I$ and that $g$ is a continuous mapping $K \to K$. Show that there is a point $p \in K$ such that $f(p) = g(p)$.

29. Prove **Baire's theorem** [ Let $X$ be a complete[8] metric space.

    (a) If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open dense subsets of $X$, then $\bigcap_{n=1}^{\infty} U_n$ is dense in $X$.

    (b) $X$ is not a countable union of nowhere dense sets.]

30. Use Baire's theorem to show the existence of $f \in C([0,1])$ that is nowhere differentiable. [Hint: Consider the sets $E_n$ of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on $f$) such that

$$|f(x) - f(x_0)| \leq n|x - x_0|$$

    for all $x \in [0,1]$. Show that $E_n$ is nowhere dense in $C([0,1])$.]

31. Prove **Banach-Steinhaus theorem** [Suppose $X$ is a Banach space and $Y$ is a normed space and that $\mathcal{A} \subset \mathcal{B}(X,Y)$. Moreover assume that

$$\sup_{T \in \mathcal{A}} \|Tx\| < \infty \quad \text{for all} \quad x \in X.$$

    Then

$$\sup_{T \in \mathcal{A}} \|T\| < \infty.]$$

32. Use Banach-Steinhaus theorem[9] to show the existence of a continuous function on $[-\pi, \pi]$ such that its Fourier series diverges at 0.

33. Prove **Perron's theorem**, i.e. prove that an $n \times n$-matrix, whose elements are all positive, has at least one positive eigenvalue and that the elements of the corresponding eigenvector are all positive.

[8] For the definition of a metric space $X$ with metric $d$ see exercise 6 in the section "normed spaces". We say that a set $A \subset X$ is open if for each $x \in A$ there is an $r > 0$ such that $\{y \in X : d(x,y) < r\} \subset A$. A set $B \subset X$ is closed if its complement $B^c$ is an open set. Given a subset $E$ of $X$. The intersection of all closed sets in $X$ containing $E$ is a closed set, is called the closure of $E$ and is denoted $\overline{E}$. The union of all open sets in $X$ contained in $E$ is an open set, is called the interior of $E$ and is denoted by $E^0$. We say that a set $E$ in $X$ is dense in $X$ if $\overline{E} = X$ and we say that $E$ is nowhere dense if $(\overline{E})^0 = \emptyset$. Finally, a metric space is called complete if for each sequence $\{x_n\} \subset X$ such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$ there exists an $x$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

[9] Hint: Let $T_n f$ denote the $n$–th partial sum of the Fourier series of $f$.

34. A linear integral operator with a positive kernel is a natural analogue of the positive matrix in Perron's theorem. Use Schauder's theorem to prove that an integral operator with positive continuous kernel has a positive eigenvalue.

35. Let $T : \bar{B}(0,1) \to \bar{B}(0,1)$ where $\bar{B}(0,1)$ is the closed unit ball in $\mathbb{R}^n$ centered at the origin. Assume that

$$|T(x) - T(y)| \leq |x - y|$$

for all $x, y \in \bar{B}(0,1)$ where $|\cdot|$ denotes the Euclidean distance. Show that $T$ has a fixed point using

   (a) Brouwder's fixed point theorem

   (b) Banach's contraction theorem[10]

36. Prove **Arzela-Ascoli's theorem** [Let $A \subset C([0,1])$. It follows that that $\overline{A}$ is compact if and only if

   (a) (uniform boundedness) there exists an $M < \infty$ such that

$$\sup_{x\in[0,1],\, f\in A} |f(x)| \leq M$$

   and

   (b) (equicontinuity) for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

   for all $x, y \in [0,1]$ with $|x - y| < \delta$ and all $f \in A$.]

37. Let $M$ be a bounded set in $C([0,1]$,not necessarily compact. Show that the set of all functions $F(x) = \int_0^x f(t)\,dt$ with $f \in M$ is relatively compact.

38. Prove **Sperner's lemma** Let $\Delta$ be a closed triangle with vertices $v_1, v_2, v_3$ and let $\tau$ be a triangulation of $\Delta$. This means that $\tau = \{\Delta_i\}_{i\in I}$ where $\Delta_i$ are closed triangles with the properties

   (a) $\Delta = \bigcup_{i\in I} \Delta_i$

   (b) For every $i, j \in I$, $i \neq j$, we have

$$\Delta_i \bigcap \Delta_j = \begin{cases} \emptyset & \text{or} \\ \text{common vertex} & \text{or} \\ \text{common side} & \end{cases}$$

Moreover let $\mathcal{V}$ denote the set of all vertices of the triangles $\Delta_i$ and let $c : \mathcal{V} \to \{1,2,3\}$ be a function that satisfies the following conditions:

   (a) $c(v_i) = i$ for $i = 1, 2, 3$

[10]Hint: Consider $T_n = (1 - \frac{1}{n})T$.

(b) $v \in \mathcal{V} \bigcap v_i v_j \in \{i, j\}$ for $i, j \in \{1, 2, 3\}$ where $v_i v_j$ denotes the line segment between $v_i$ and $v_j$.

Then there exists a triangle $\Delta_i$ such that the vertices of the triangle take different values.

39. Prove Brouwer's fixed point theorem in a special case[11] (n=2): Let $T : K \to K$ be a continuous mapping where $K$ denotes the set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \Sigma_{i=1}^{3} x_i = 1,\ x_i \geq 0 \text{ all } i\}$. Then $T$ has a fixed point.

[11]Consider a sequence of finer and finer triangulations of $K$ and make use of the function $c : K \to \{1, 2, 3\}$ defined by

$$c(\mathbb{x}) = \min\{i : (T(\mathbb{x}))_i < x_i\}$$

where $\mathbb{x} = (x_1, x_2, x_3)$. Note that the function $c$ is well-defined provided $T$ has no fixed point, and apply Sperner's lemma.

## 1.5 Hilbert spaces

*Key words:* inner product, inner product space, polarization identity, Hilbert space, orthogonality, strong/weak convergence, orthonormal sequence, Gram–Schmidt orthonormalization process, complete sequence, orthogonal complement, convex set, orthogonal projection and decomposition, separable Hilbert space

1. Let $z_1, \ldots, z_n$ be complex numbers. Show that
$$|z_1 + \ldots + z_n| \leq \sqrt{n}\|(z_1, \ldots, z_n)\|.$$

2. Let $x, y$ be vectors in a complex vector space with inner product, and assume that
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$
Does this imply that $\langle x, y\rangle = 0$?

3. Let $H$ be a Hilbert space. Show that
$$\|x - z\| = \|x - y\| + \|y - z\|$$
if and only if $y = \alpha x + (1 - \alpha)z$ for some $\alpha \in [0, 1]$.

4. Let $\|\cdot\|$ denote the norm in a Hilbert space. Prove that
$$\|x + y\|\,\|x - y\| \leq \|x\|^2 + \|y\|^2$$
and
$$\|x + y\|^2 - \|x - y\|^2 \leq 4\|x\|\,\|y\|.$$

5. Let $E$ be an inner product space. Show that for $x, y \in E$, $x \perp y$ if and only if $\|\alpha x + \beta y\|^2 = \|\alpha x\|^2 + \|\beta y\|^2$ for all scalars $\alpha$ and $\beta$.

6. Show that $C([0, 1])$ (equipped with the sup-norm) is not an inner product space.

7. Prove that any complex Banach space with norm $\|\cdot\|$ satisfying the parallelogram law is a Hilbert space with the inner product
$$\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2],$$
and $\|x\|^2 = \langle x, x\rangle$.

8. Let $T : E \to E$ be a bounded linear operator on a complex inner product space. Show that $T = 0$ if $\langle Tx, x\rangle = 0$ for all $x \in E$. Show that this does not hold in the case of real inner product spaces.

9. Suppose $x_n \to x$ and $y_n \to y$ in a Hilbert space $H$ and $\alpha_n \to \alpha$ in $\mathbb{C}$. Prove that

(a) $x_n + y_n \to x + y$
(b) $\alpha_n x_n \to \alpha x$
(c) $\langle x_n, y_n \rangle \to \langle x, y \rangle$
(d) $\|x_n\| \to \|x\|$

10. Suppose $x_n \stackrel{w}{\to} x$ and $y_n \stackrel{w}{\to} y$ in a Hilbert space $H$ and $\alpha_n \to \alpha$ in $\mathbb{C}$. Prove or disprove that

    (a) $x_n + y_n \stackrel{w}{\to} x + y$
    (b) $\alpha_n x_n \stackrel{w}{\to} \alpha x$
    (c) $\langle x_n, y_n \rangle \to \langle x, y \rangle$
    (d) $\|x_n\| \to \|x\|$

11. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for $H$. Assume that the sequence $(f_n)_{n=1}^{\infty}$ in $H$ satisfies the conditions $\|f_n\| = 1$ and $f_n \in \{e_1, e_2, \ldots, e_n\}^{\perp}$ for $n = 1, 2, \ldots$. Show that $f_n \stackrel{w}{\to} \mathbf{0}$.

12. Suppose $x_n \stackrel{w}{\to} x$ in a Hilbert space $H$. Show[12] that there is a positive constant $M$ such that
$$\sup_n \|x_n\| \leq M.$$

13. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence, i.e. $\sup_n \|x_n\| \leq M$, in a separable Hilbert space $H$. Show that there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ and an $x \in H$ such that
$$x_{n_k} \stackrel{w}{\to} x.$$
What happens if $H$ is not separable?

14. Suppose $x_n \stackrel{w}{\to} x$ in a Hilbert space $H$. Show that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that
$$\frac{1}{m}\Sigma_{k=1}^{m} x_{n_k} \to x \ \text{ i } \ H,$$
då $m \to \infty$.

15. Consider $\mathbb{R}^n$ as a Hilbert space with the standard inner product and the corresponding norm, i.e. the Euclidean metric. Assume that $S$ is a closed convex set in $\mathbb{R}^n$ and that for each $x \in \mathbb{R}^n$ there exists a unique $y \in S$ such that
$$\|x - y\| = \sup_{z \in S} \|x - z\|.$$
Show that $S$ consists of a single element.

16. Assume that $M$ is a closed subspace of a Hilbert space $H$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to $x$ in $H$. Moreover let $x_n = y_n + z_n$, $n = 1, 2, \ldots$, be the orthogonal decomposition of $x_n$ with $y_n \in M$ and $z_n \in M^{\perp}$. Show that $y_n$ converges to $y$ and $z_n$ converges to $z$ where $x = y + z$ is the orthogonal decomposition of $x$.

[12] Hint: Use Banach-Steinhaus theorem above

17. What is the orthogonal complement of all even functions in $L^2([-1,1])$?

18. Let $M$ be the subset $\{(x_n)_{n=1}^{\infty} : x_{2n} = 0 \text{ for all } n \in \mathbb{Z}_+\}$ in $l^2$. Give $M^{\perp}$ and $M^{\perp\perp}$.

19. Let $A$ be a subset of an inner product space. Show that

$$A^{\perp\perp} = \overline{\mathrm{Span} A}.$$

20. Let $A$ and $B$, $\emptyset \neq A \subset B$, be subsets of an inner product space. Show that

    (a) $B^{\perp} \subset A^{\perp}$

    (b) $A^{\perp\perp\perp} = A^{\perp}$.

21. Let $M \neq \emptyset$ be a subset of a Hilbert space $H$. Show that $\mathrm{Span}\, M$ is dense in $H$ if and only if $M^{\perp} = \{0\}$. By the span of a set $\mathcal{A}$ we mean all finite linear combinations of the elements in the set $\mathcal{A}$.

22. Let $M \neq \emptyset$ be any subset of a Hilbert space $H$. Show that $M^{\perp\perp}$ is the smallest closed subspace in $H$ that contains $M$.

23. Let $(x_n)_{n=1}^{\infty}$ be a complete orthonormal sequence in a Hilbert space $H$. Show that

$$\langle x, y\rangle = \Sigma_{n=1}^{\infty} \langle x, x_n\rangle \overline{\langle y, x_n\rangle}$$

for all $x, y \in H$. Also show that the reverse implication is true.

24. Let $(x_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $H$. Show that $(x_n)_{n=1}^{\infty}$ is complete if and only if the closure of the span of $(x_n)_{n=1}^{\infty}$ equals $H$.

25. If $(x_n)_{n=1}^{\infty}$ is a complete orthonormal set for a vector subspace $S$ of a Hilbert space $H$, then any $x \in S$ can be expressed in the form $x = \Sigma c_n x_n$. Conversely, if $y = \Sigma c_n x_n$, does if follow that $y \in S$? What happens if $S$ is a Hilbert subspace of $H$?

26. Given a convergent infinite series, one cannot in general rearrange the terms; if the sequence $(v_n)$ is a rearrangement of a series $(u_n)$, and $\Sigma u_n = U$, then $\Sigma v_n$ need not equal $U$, unless $\Sigma u_n$ converges absolutely. However, prove that if $(e_n)$ is a complete orthonormal set and $(f_n)$ is a sequence obtained by arranging $(e_n)$ in a different order, then $(f_n)$ is a complete orthonormal set, and therefore the series $x = \Sigma\langle x, e_n\rangle e_n$ can be rearranged.

27. (A space with no complete ON sequence) The set of all periodic functions $\mathbb{R} \to \mathbb{C}$ is clearly not a vector space. But if we consider the set $M$ of functions which are sums and products of finitely many periodic functions, we obtain a vector space. The elements of $M$ are called **almost-periodic functions**. It can be proved that for any $f, g \in M$,

$$\lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} f(t)\overline{g(t)}\, dt\right)$$

exists and defines an inner product on $M$. Verify that any two members of the family of functions $e^{iat}$, where $a$ is real, are orthogonal in the inner product space $M$. Deduce that $M$ has no countable basis.

28. Find an orthonormal basis of the subspace $\mathrm{Span}\{1+x, 1-x\}$ of $L^2([0,1])$.

29. Let $P$ and $Q$ denote orthogonal projections onto two subspaces in a Hilbert space. Prove that $\|P-Q\| \leq 1$.

30. Suppose $S$ is a closed convex subset of a Hilbert space $H$ and let $P_S$ denote the orthogonal projection onto $S$, i.e. for any $x \in H$, $P_S(x)$ denotes the point in $S$, which is nearest to $x$. Prove that

$$\|P_S(x) - P_S(y)\| \leq \|x-y\| \text{ for all } x, y \in H.$$

31. In the vector space $\mathbb{R}^n$ use the norm $\|u\| = \Sigma|u_i|$. Let $x = (1, -1, 0, \ldots, 0)$ and let $E$ be the subspace $\{(t, t, 0, \ldots, 0) : t \in \mathbb{R}\}$. Setting $y_t = (t, t, 0, \ldots, 0)$ for the elements of $E$, show that all $y_t$ with $|t| \leq 1$ have the same distance from $x$, and are closer to $x$ than any $y_t$ with $|t| > 1$. This shows that the best approximation in a subspace can be non-unique in normed spaces, though in Hilbert spaces they are unique. Deduce that the norm $\Sigma|u_i|$ cannot be obtained from any inner product.

32. Let $H = \{f \in L^2([0,1]) : f' \in L^2([0,1])\}$, and for $f, g \in H$ define

$$\langle f, g\rangle = f(0)g(0) + \int_0^1 f'(s)g'(s)\,ds.$$

Take $L^2$ here to be the space of real functions. Show that $H$ is a Hilbert space. For each $t \in [0,1]$ define a function $R_t \in H$ by $R_t(s) = 1 + \min(s,t)$, where $\min(s,t)$ denotes the smaller of $s$ and $t$. Show that $\langle f, R_t\rangle = f(t)$ for all $f \in H$.

Now consider the following problem in approximation theory. The interval $[0,1]$ is divided into subintervals given by numbers $0 = t_1 < t_2 < \ldots < t_n = 1$. Given a function $f$, we wish to approximate it by a piecewise linear function $F$ which is linear in each subinterval. Show that the set of all such functions $F$ is the subspace spanned by $\{R_{t_i} : i = 1, 2, \ldots, n\}$. Show that the best piecewise linear approximation to $f$ in the sense of the norm corresponding to the above inner product in $H$ is the piecewise linear function $F$ which equals $f$ at the points $t_i$.

33. Suppose $A : H \to H$ is a linear mapping that satisfies

$$\langle Ax, y\rangle = \langle x, Ay\rangle \text{ all } x, y \in H.$$

Prove that $A$ is a continuous mapping.[13]

[13] Hint: Apply Banach–Steinhaus theorem

## 1.6 Linear operators on Hilbert spaces

*Key words:* bilinear functional, quadratic form, coercive functional, adjoint operator, self-adjoint operator, inverse operator, normal operator, isometric operator, unitary operator, positive operator, projection operator, compact operator, finite-dimensional operator, eigenvalues/eigenvectors, resolvent, spectrum, unbounded operators

1. Let $A$ be a self-adjoint operator on a Hilbert space $H$ and assume that $\overline{\mathcal{R}(A)} = H$. Show that $A : H \to \mathcal{R}(A)$ is an invertible mapping.

2. Assume that $A_n \to A$ in $\mathcal{B}(H, H)$, where $H$ is a Hilbert space. Show that $A$ is self-adjoint if all $A_n$ are self-adjoint.

3. Let $A$ be a linear compact operator on a Hilbert space $H$. Prove that $I + A$ is a compact operator if and only if $H$ is finite-dimensional.

4. Let $B$ be a bounded linear operator on a Hilbert space. Prove that
$$\mathcal{R}(B)^{\perp} = \mathcal{N}(B^*)$$
and
$$\overline{\mathcal{R}(B)} = \mathcal{N}(B^*)^{\perp}.$$

5. Let $A$ be a compact linear operator on a Hilbert space $H$. Prove that $\mathcal{R}(I - A)$ is a closed subspace[14] of $H$.

6. Let $A$ be a compact linear operator on a Hilbert space. Prove that
$$\mathcal{R}(I - A) = \mathcal{N}(I - A^*)^{\perp}.$$

7. Assume that $x_n \xrightarrow{w} x$ in a Hilbert space $H$. Moreover assume that $A : H \to H$ is a bounded linear mapping. Does it follow that $Ax_n \xrightarrow{w} Ax$?

8. Show that for every compact operator $A$ on a Hilbert space $H$ there exists a sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{B}(H, H)$ such that $\dim \mathcal{R}(A_n) < \infty$ for $n = 1, 2, \ldots$ and $A_n \to A$ in $\mathcal{B}(H, H)$.

9. Show that the integral operator on $L^2([0, 1])$ with kernel $K$ satisfying
$$\int_0^1 \int_0^1 |K(x, y)|^2 \, dx dy < \infty$$
is compact[15].

[14]Hint: Let $y \in H$ and suppose $x_n - Ax_n \to y$. Show that one can pick $x_n$ to belong to $\mathcal{N}(I - A)^{\perp}$ for every $n$. Show that $\{x_n\}$ must be bounded.

[15]Hint: Approximate $K$ by $\tilde{K}(x, y) = \Sigma_{i,j=1}^{n} p_i(x) q_j(y)$. Alternatively approximate $K$ by continuous $\tilde{K}$ and use Arzela-Ascoli's theorem.

10. (a) Suppose $f \in L^1(\mathbb{R})$ and set

$$(Ax)(t) = \int_{-\infty}^{\infty} x(s) f(t-s)\, ds \;\; x \in L^2(\mathbb{R}).$$

Prove that $A$ defines a bounded linear operator on $L^2(\mathbb{R})$ with an operator norm $\leq \|f\|_1$.

(b) Suppose $h > 0$ and set

$$(Bx)(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(s)\, ds \;\; x \in L^2(\mathbb{R}).$$

Prove that $B$ defines a bounded linear operator on $L^2(\mathbb{R})$ with norm 1.

11. Set

$$(Ax)(t) = tx(t), \;\; x \in L^2([0,1]).$$

Prove that $A$ defines a linear bounded self-adjoint operator on $L^2([0,1])$ without eigenfunctions.

12. Find[16] a mapping $f : [0,1] \to L^2([0,1])$ such that $f(t_1) \neq f(t_2)$ for all $t_1 \neq t_2$ and such that the vectors $f(t_1) - f(t_2)$ and $f(t_3) - f(t_4)$ are orthogonal for all $t_1 < t_2 < t_3 < t_4$.

13. The operator $A$ on $L^2([0,1])$ is defined by

$$(Af)(x) = \int_0^x f(y)\, dy, \;\; 0 \leq x \leq 1.$$

Find $A^*$.

14. Show that an operator of rank $n$ can have at most $n$ eigenvalues.

15. Set

$$(Ax)(t) = \int_{-\infty}^{\infty} \frac{x(s)}{1+(t-s)^2}\, ds, \;\; x \in L^2(\mathbb{R}).$$

Prove that $A$ defines a linear bounded and self-adjoint operator on $L^2(\mathbb{R})$. Finally prove that $A$ is not a compact operator.

16. Set

$$(Tf)(x) = \int_0^{\pi} \sin(x+y) f(y)\, dy, \;\; 0 \leq x \leq \pi.$$

Find the norm of $T$ regarded as an operator on

(a) the Banach space $C([0,\pi])$

(b) the Hilbert space $L^2([0,\pi])$.

17. Give an example of a non-self-adjoint operator on a Hilbert space $H$ whose range is $H$ and which is not invertible.

[16] Hint: Let $f(t)$ be the characteristic function for the set $[0,t]$ for $t \in [0,1]$.

18. Let $T_n : E \to H$, $n = 1, 2, \dots$, be a sequence of bounded linear operators from a normed space $E$ into a Hilbert space $H$. We say that

   (a) $(T_n)_{n=1}^{\infty}$ is convergent in $\mathcal{B}(E, H)$ (or convergent in norm in $\mathcal{B}(E, H)$ or uniformly operator convergent) if $(T_n)_{n=1}^{\infty}$ is convergent in $\mathcal{B}(E, H)$;

   (b) $(T_n)_{n=1}^{\infty}$ is strongly operator convergent if $(T_n(x))_{n=1}^{\infty}$ converges in $H$ for all $x \in E$;

   (c) $(T_n)_{n=1}^{\infty}$ is weakly operator convergent if $(T_n(x))_{n=1}^{\infty}$ converges weakly in $H$ for all $x \in E$.

   Show that a) $\Rightarrow$ b) $\Rightarrow$ c). Moreover, let $A_n, B_n$ be operators on $l^2$ defined by

$$A_n((x_1, x_2, \dots)) = (\underbrace{0, \dots, 0}_{n \text{ positions}}, x_{n+1}, x_{n+2}, \dots)$$

   and

$$B_n((x_1, x_2, \dots)) = (\underbrace{0, \dots, 0}_{n \text{ positions}}, x_1, x_2, \dots).$$

   In what modes do these sequences of operators converge?

19. A bounded linear operator $A$ on a Hilbert space $H$ is called **unitary** if $A^*A = AA^* = I$. Show that if $A$ is unitary then $\|Ax\| = \|x\|$ for all $x \in H$, i.e. unitary operators do not change lengths. Deduce that all eigenvalues of unitary operators have modulus 1, and eigenvectors belonging to different eigenvalues are orthogonal. Show that all unitary operators are invertible.

   If $B$ is a self-adjoint operator, show that $e^{iB}$ is unitary.

20. A bounded linear operator $A$ on a Hilbert space $H$ is called a **Hilbert-Schmidt operator** if the series $\Sigma_{ij}|\langle Ae_i, f_j\rangle|^2$ converges whenever $(e_i)$ and $(f_j)$ are orthonormal bases for the Hilbert space $H$. Show that this sum equals $\Sigma_i\|Ae_i\|^2$, and deduce that it is independent of the choice of bases $(e_i)$ and $(f_j)$.

   Show that the set of Hilbert-Schmidt operators on a given Hilbert space $H$ is a vector space, and that $\|A\|_{HS} = (\Sigma_i\|Ae_i\|^2)^{1/2}$ is a norm on that space. Show that $\|A\|_{HS} \geq \|A\|$ where $\|A\|$ is the usual operator norm. Give an example in which $\|A\|_{HS} > \|A\|$.

   If $A$ and $B$ are Hilbert-Schmidt operators, show that $\Sigma\langle Ae_i, Be_i\rangle$ converges absolutely for every orthonormal basis $(e_i)$, and is independent of the choice of $(e_i)$. Show that one can define an inner product $[A, B]$ on the space of Hilbert-Schmidt operators on $H$ by $[A, B] = \Sigma\langle Ae_i, Be_i\rangle$.

   If $A$ and $B$ are integral operators on $L^2([0, 1])$ with continuous kernels $K$ and $L$ respectively, show that they are Hilbert-Schmidt operators, and $[A, B] = \int\int K(s, t)\overline{L(s, t)}\, dsdt$.

21. A bounded linear operator $A$ on a Hilbert space is called **normal** if it commutes with its adjoint, $AA^* = A^*A$. Every self-adjoint operator is obviously normal.

(a) Show that if the function $K(x,y)$ satisfies $K(x,y) = \overline{K(y,x)}$, then for any real $d$, the operator $u \mapsto du + i\int_0^1 K(x,y)u(y)\,dy$ on the complex Hilbert space $L^2([0,1])$ is normal.

(b) Show that if $B, C$ are commuting self-adjoint operators, then $B + iC$ is normal.

(c) Prove the converse of (b), i.e. for any normal operator $A$, there are self-adjoint commuting operators $B, C$ such that $A = B + iC$.

22. Show that a compact normal operator has a complete set of orthogonal eigenvectors.

23. Given an infinite matrix of numbers $k_{ij}$, $i, j = 1, 2\ldots$, we say that the double series $\Sigma_{ij}|k_{ij}|^2$ converges if for each $i$ the series $\Sigma_j|k_{ij}|^2$ converges to a number $L_i$ such that $\Sigma_i L_i$ converges, and for each $j$ the series $\Sigma_i|k_{ij}|^2$ converges to a number $M_j$ such that $\Sigma_j M_j$ converges. If $\Sigma_{ij}|k_{ij}|^2$ converges and $k_{ij} = \overline{k_{ji}}$ for all $i, j$, we define an operator $K$ on the space $l^2$ by $(Kx)_i = \Sigma_{j=1}^{\infty} k_{ij}x_j$. Show that $K$ is a compact self-adjoint operator $l^2 \to l^2$, and write out what the spectral theorem says in this case.

24. Let $(p_i)$ and $(q_i)$ be two complete orthonormal sets for $L^2([0,1])$. Let $H$ be the space of square-integrable functions of two variables on the square $0 \le x, y \le 1$, with inner product $\int_0^1\int_0^1 f(x,y)\overline{g(x,y)}\,dxdy$.

(a) Show that the set of functions $p_i(x)q_j(y)$ is orthonormal in $H$.

(b) Show that if $\phi \in H$ and $\int_0^1\int_0^1 \phi(x,y)p_i(x)q_j(y)\,dxdy = 0$ for all $i, j$, then $\phi = 0$.

(c) The set of functions $p_i(x)q_j(y)$ is labelled by two integers and is therefore countable, and can be arranged in a sequence. Prove that this sequence is a complete orthonormal sequence.

25. Given a function $K$ such that $K(x,y) = \overline{K(y,x)}$ and $\int_0^1\int_0^1 |K(x,y)|^2\,dxdy$ exists, let $\lambda_i$ and $\phi_i$ be the eigenvalues and orthonormal eigenfunctions of the integral operator on $L^2([0,1])$ whose kernel is $K$. Show that

$$K(x,y) = \Sigma_i \lambda_i \phi_i(x)\overline{\phi_i(y)},$$

the convergence being with respect to the norm in the space $H$ in the previous problem. Show also that

$$\int_0^1\int_0^1 |K(x,y)|^2\,dxdy = \Sigma_i|\lambda_i|^2.$$

26. $K : \mathbb{R}^2 \to \mathbb{C}$ is a piecewise continuous function, and $K(x,y) = \overline{K(y,x)}$. The integral operator $A$ on $L^2([0,1])$ with kernel $K$ has eigenvalues $\lambda_i$ and orthonormal eigenfunctions $\phi_i$.

(a) Show that the series $\Sigma c_n\phi_n(x)$ converges absolutely and uniformly if the constants satisfy $\Sigma|c_n/\lambda_n|^2 < \infty$.

(b) Show that if $f$ is in the range of $A$, then the series $\Sigma\langle f, \phi_n\rangle\phi_n(x)$ converges absolutely and uniformly to $f$ on $[0,1]$. Is this still true if we remove the condition that $f$ lies in the range of $A$?

27. Above it was shown that the eigenvalues $\lambda_i$ of an integral operator with square-integrable kernel are such that $\Sigma|\lambda_i|^2$ converges. Is this true for compact self-adjoint operators in general?

## 1.7 Ordinary differential equations

*Key words:* Green's function, symmetric operators

1. Calculate the Green's functions for the boundary value problems

   a) $\begin{cases} u''(x) + u(x) = f(x) \\ u'(0) = u'\left(\frac{\pi}{2}\right) = 0,\ 0 \le x \le \frac{\pi}{2} \end{cases}$

   b) $\begin{cases} u''(x) = f(x) \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0,\ 0 \le x \le 1 \end{cases}$

   c) $\begin{cases} u''(x) + u(x) = f(x) \\ u(0) = u'(0) = 0,\ 0 \le x \le T \end{cases}$

   d) $\begin{cases} \frac{1}{6}u^{(4)}(x) = f(x) \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{cases}$

   e) $\begin{cases} u^{(4)}(x) = f(x) \\ u(0) = u''(0) = u(1) = u''(1) = 0 \end{cases}$

2. Show that (using the notations from "A note on ordinary differential equations") the boundary value problem

$$\begin{cases} Lu = f \\ Ru = c \end{cases}$$

   is uniquely solvable for every $f \in C^n(I)$ and $c \in \mathbb{C}^n$ iff

$$\det\{R_j u_k\}_{1 \le j,k \le n} \ne 0.$$

3. Show that the Green's function $g(x,t)$ in Example 1 on page 9 in "A note on ordinary differential equations" satisfies $g(x,t) = \overline{g(t,x)}$ and hence the operator $\tilde{G} : L^2([0,1]) \to L^2([0,1])$ defined by

$$(\tilde{G}f)(x) = \int_0^1 g(x,t)f(t)dt,$$

   is self-adjoint.

4. Show that the problem

$$\begin{cases} u''(x) + u(x) = e^{ix} + \frac{1}{2}\mathrm{Re}\ u(x),\ 0 \le x \le \pi/2 \\ u'(0) = u'(\pi/2) = 0,\ u \in C^2([0,\pi/2]) \end{cases}$$

   has a unique solution.

5. Set $(Lu)(x) = u^{(4)}(x)$, $0 \le x \le 1$. Show that $L_0$ is symmetric if

   (a) $R_1u = u(0), R_2u = u'(0), R_3u = u(1), R_4u = u'(1)$

   (b) $R_1u = u(0), R_2u = u''(0), R_3u = u(1), R_4u = u''(1)$.

6. Assume that $(Lu)(x) = -u''(x) + u(x)$, $0 \le x \le 1$ and that $R_1 u = u(0) - u(1)$ and $R_2 u = u'(0) - u'(1)$. Show that

   (a) $L_0$ is bijective

   (b) $L_0$ has both 1- and 2-dimensional eigenspaces.

7. Assume that $(Lu)(x) = (p(x)u'(x))' - q(x)u(x)$, $a \le x \le b$, where $p \in C^1(I)$ and $q \in C(I)$ are real-valued and $p(x) > 0$, $a \le x \le b$. Moreover assume that

$$R_1 u = \alpha_{11} u(a) + \alpha_{21} u'(a)$$

and

$$R_2 u = \beta_{12} u(b) + \beta_{22} u'(b)$$

where $(\alpha_{11}, \alpha_{21}) \in \mathbb{R}^2 \setminus \{0\}$ and $(\beta_{12}, \beta_{22}) \in \mathbb{R}^2 \setminus \{0\}$. Show that $L_0$ is symmetric.

8. Assume that the integral operator

$$(Qf)(x) = \int_a^b q(x,y) f(y) dy, \; a \le x \le b,$$

defined on $L^2(I)$ with an $L^2$- kernel $q$ is self-adjoint and has the eigenvalues $(\lambda_i)_1^\infty$, counted with multiplicity, and corresponding eigenfunctions $(e_i)_1^\infty$.

   (a) Use Bessel's inequality to show that

$$\sum_1^\infty \lambda_i^2 |e_i(x)|^2 \le \int_a^b |q(x,y)|^2 dy.$$

   (b) Show that

$$\sum_1^\infty \lambda_i^2 \le \int_a^b \int_a^b |q(x,y)|^2 dx dy.$$

   (c) Show that

$$q(x,y) = \sum_1^\infty \lambda_i e_i(x) \overline{e_i(y)} \quad \text{i } L^2(I \times I).$$

9. Prove that

$$\min(x,y) = \sum_{n=0}^\infty \frac{2}{(n+\frac{1}{2})^2 \pi^2} \sin\left(n + \frac{1}{2}\right)\pi x \sin\left(n + \frac{1}{2}\right)\pi y$$

in $L^2([0,1] \times [0,1])$.

10. Show that the series in Theorem 1.7 in "A note on ordinary differential equations" converges uniformly to $u$.

11. Prove that there is no function $u$ defined in the interval $[0,1]$ such that

$$\begin{cases} xu'(x) + u(x) = 0, & 0 \le x \le 1 \\ u(0) = 1. \end{cases}$$

12. Prove the existence of solutions $u$ of the following boundary value problem

$$\begin{cases} -u''(x) = 3(1 + u^2(x)), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

13. Prove the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} -u''(x) = 7\dfrac{u(x)}{1 + u^2(x)} + \sin(\pi x), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

14. Prove the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} 4u''(x) = |x + u(x)|, & 0 \le x \le 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, & u \in C^2([0,1]). \end{cases}$$

15. Let $\lambda \in \mathbb{R}$ be different from 0.

    (a) Solve the equation

    $$\begin{cases} |u'(x)|^2 + \dfrac{1}{\lambda}u''(x) = 1, & 0 \le x \le 1 \\ u(-1) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

    (b) Let $u(x) = u(x,\lambda)$ be the solution in part (a). Calculate $\lim_{\lambda \to \pm\infty} u(x,\lambda)$.

## 1.8 Calculus of variation

*Key words:* Gateaux derivative, Fréchet derivative, convex functions, stationary point, Euler-Lagrange equation, variational problems with constraints

1. Show that if $h \in C([a,b])$ and $\int_a^b h(x)v'(x)\,dx = 0$ for all $v \in C^1([a,b]) \bigcap \{v(a) = v(b) = 0\}$ then $h =$ constant on $[a,b]$.

2. Let $m \in \mathbb{N}$. Show that if $g \in C([a,b])$ and $\int_a^b g(x)v(x)\,dx = 0$ for all $v \in C^m([a,b]) \bigcap \{v^{(k)}(a) = v^{(k)}(b) = 0 \ k = 0,1,\ldots,m\}$ then $g = 0$ on $[a,b]$.

3. Set $E = C^1([a,b])$ and let $\delta I(y;v)$ denote the **(Gateaux–)variation**

$$dI(y;v) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(I(y+\epsilon v) - I(y)),$$

   where $y, v \in E$. Calculate $\delta I(y;v)$ for

   (a) $I(y) = \int_a^b ((y(x))^3 + x(y'(x))^2)\,dx$

   (b) $I(y) = \int_a^b (e^x y(x) - 3(y'(x))^4)\,dx + 2(y'(a))^2$

   (c) $I(y) = \int_a^b y(x)\,dx / \int_a^b (1 + (y'(x))^2)\,dx$

4. Give the Euler-Lagrange equation for $F(x,y,z) = 2xy - y^2 + 3zy^2$, $(x,y,z) \in \mathbb{R}^3$. Find the stationary solutions for $F$ above on $D = C^1([0,1]) \bigcap \{y(0) = 0, y(1) = 1\}$.

5. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Show that

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for all } x, y \in \mathbb{R}^n,\ t \in (0,1),$$

   i.e. $f$ is a **convex function**, if and only if

$$f(x) \ge f(y) + \nabla f(y) \cdot (x - y)) \quad \text{for all } x, y \in \mathbb{R}^n. \tag{3}$$

   Moreover interpret (3) geometrically.

6. Let $I : D \to \mathbb{R}$ be a functional defined on a subset $D$ of a vector space. We say that $I$ is **convex on** $D$ if

$$I(y+v) - I(y) \ge \delta I(y;v) \quad \text{for all } y, v \in D.$$

   Now set $D = C^1([a,b]) \bigcap \{y(a) = \alpha,\ y(b) = \beta\}$ for $\alpha, \beta \in \mathbb{R}$. Moreover, assume that $F \in C^2([a,b] \times \mathbb{R}^2)$ and that $F(x,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is a convex function for all $x \in [a,b]$. Show that $I(y) = \int_a^b F(x, y(x), y'(x))\,dx$ is convex on $D$ and that $y_0 \in D$ is a minimizer on $D$ provided

$$\frac{d}{dx} F'_{y'} F(x, y_0(x), y_0'(x)) = F'_y(x, y_0(x), y_0'(x)) \quad \text{for } x \in (a,b).$$

7. Let $D$ be a subset of a vector space and let $I, G_1, \ldots, G_N$ be functionals defined on $D$. Show that if there are some constants $\lambda_1, \ldots, \lambda_N$ and a vector $y_0 \in D$ such that $y_0$ is a minimizer of $\tilde{I} \equiv I + \lambda_1 G_1 + \ldots + \lambda_N G_N$ on $D$ then $y_0$ is a minimizer of $I$ on $D \bigcap \{y \in D : G_j(y) = G_j(y_0),\ j = 1, \ldots, N\}$.

8. Minimize $I(y) = \int_0^1 (y'(x))^2\, dx$ on $D = C^1([0,1]) \bigcap \{y(0) = y(1) = 0\}$ when restricted to the set $\{y \in C^1([0,1]) : G(y) \equiv \int_0^1 (y(x))^2\, dx = 1\}$.

9. Consider the minimizing problem

$$\inf_{y \in D} I_F(y),$$

where $D = C^2([a,b]) \bigcap \{y(a) = \alpha_1, y'(a) = \alpha_2, y(b) = \beta_1, y'(b) = \beta_2\}$ for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $F \in C^3([a,b] \times \mathbb{R}^3)$ and

$$I_F(y) = \int_a^b F(x, y(x), y'(x), y''(x))\, dx.$$

Give a necessary condition (= Euler-Lagrange equation) on $y_0$ to be a minimizer on $D$.

10. Consider the functional $I : C^1([a,b]) \bigcap \{y(a) = \alpha, y(b) = \beta\} \to \mathbb{R}$, where $I(y) = \int_a^b F(y(x), y'(x))\, dx$ and $\alpha, \beta \in \mathbb{R}$. Assume that $y_0$ is a minimizer on $D$. Show that

$$F(y_0(x), y_0'(x)) - y_0'(x) F'_{y'}(y_0(x), y_0'(x)) = \text{constant}$$

on $(a,b)$.