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Mathematics
Chalmers & GU

Second Assignment

hand in at latest on Friday May 9

1. Let x and y be elements in a complex vector space with inner product and assume that
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$
Does this imply that $\langle x, y\rangle = 0$? What happens if the complex vector space is replaced by a real vector space?

2. The operator A on $L^2([0,1])$ is defined by
$$(Af)(x) = \int_0^x f(y)\, dy, \quad 0 \leq x \leq 1.$$
Find A^*.

3. Set
$$(Af)(t) = \int_{-\infty}^{\infty} \frac{f(s)}{1+(t-s)^2}\, ds, \quad f \in L^2(\mathbb{R}).$$
Prove that A defines a linear bounded and self-adjoint operator on $L^2(\mathbb{R})$. Finally prove that A is not a compact operator.

4. Suppose S is a closed convex subset of a Hilbert space H and let P_S denote the orthogonal projection onto S, i.e. for any $x \in H$, $P_S(x)$ denotes the point in S, which is nearest to x. Prove that
$$\|P_S(x) - P_S(y)\| \leq \|x-y\| \text{ for all } x, y \in H.$$

5. The operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is defined by the equation
$$Tu(x) = u(x) + \int_0^x u(s)\, ds, \quad 0 \leq x \leq 1.$$
Prove that $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{C}[0,1]$. Finally determine the inverse T^{-1} of T and show that T^{-1} is a bounded operator.