TMA 401/MAN 670 Functional Analysis 2002/2003
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Third Assignment

hand in at latest on Friday May 23

1. Prove the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} -u''(x) = 7\dfrac{u(x)}{1+u^2(x)} + \sin(\pi x), & 0 \le x \le 1 \\ u(0) = u(1) = 0, & u \in C^2([0,1]). \end{cases}$$

2. Prove that there is no function u defined in the interval $[0,1]$ such that

$$\begin{cases} xu'(x) + u(x) = 0, & 0 \le x \le 1 \\ u(0) = 1. \end{cases}$$

3. Show that (using the notations from "A note on ordinary differential equations") the boundary value problem

$$\begin{cases} Lu = f \\ Ru = c \end{cases}$$

is uniquely solvable for every $f \in C^n(I)$ and $c \in \mathbb{C}^n$ iff

$$\det\{R_j u_k\}_{1 \le j,k \le n} \ne 0.$$

4. Let A be a compact linear operator on a Hilbert space H. Prove that $\mathcal{R}(I - A)$ is a closed subspace of H.

5. Calculate

$$||A^3 - 2A^2 - 4A + 4I||$$

where $A : H \to H$ is a positive compact self-adjoint operator with $||A|| = 1$ on a Hilbert space H.