Förslag till lösningar till tentamen i TMA401/MAN670, 2003-05-31

Problem 1

We will prove that

$$\begin{cases} u''(x) + u'(x) = \arctan u(x^2), & x \in (0,1) \\ u(0) = u(1) = 0 \\ u \in C^2 \end{cases}$$

has a unique solution.

1. Determine the Green's function for $\begin{cases} u'' + u' = F & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$:

   Set $e(x,t) = a_1(t) + a_2(t)e^{-x}$ where $\begin{cases} e(t,t) = 0 \\ e'_x(t,t) = 1 \end{cases}$. This gives $e(x,t) = 1 - e^{t-x}$. The Green's function takes the form

   $$g(x,t) = \theta(x-t)(1 - e^{t-x}) + b_1(t) + b_2(t)e^{-x}.$$

   Here $g(0,t) = g(1,t) = 0$, $0 < t < 1$ implies

   $$\begin{cases} b_1(t) = \frac{e^t - 1}{e-1} \\ b_2(t) = -\frac{e^t-1}{e-1}, \end{cases}$$

   Hence $g(x,t) = \theta(x-t)(1-e^{t-x}) + \frac{e^t-1}{e-1}(1-e^{-x})$. We see (a simple calculation) that $g(x,t) \leq 0$ for all x, t.

2. Set

   $$\begin{cases} (Tu)(x) = \int_0^1 g(x,t) \arctan u(t^2)\, dt, & 0 \leq x \leq 1 \\ u \in C[0,1] \end{cases}$$

   The boundary value problem has a unique solution iff T has a unique fixed point. For $u, v \in C[0,1]$ we obtain

   $$\begin{aligned} |(Tu)(x) - (Tv)(x)| &= \int_0^1 |g(x,t)| |\arctan u(t^2) - \arctan v(t^2)|\, dt \leq \\ &\leq \{\text{mean value theorem}\} \leq \\ &\leq \int_0^1 |g(x,t)||u(t^2) - v(t^2)|\, dt \leq \\ &\leq \int_0^1 (-g(x,t))\, dt \|u - v\|_\infty. \end{aligned}$$

Vi note that (small calculation)

$$\int_0^1 (-g(x,t)\,dt = \max_{0\le x\le 1}(-x+\frac{e}{e-1}(1-e^{-x}) \equiv c < 1$$

and hence

$$\|Tu-Tv\|_\infty \le c\|u-v\|_\infty.$$

This shows that T is a contraction on the Banach space $C[0,1]$ and the conclusion follows from Banach's fixed point theorem.

Problem 2

Let T, H, $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ be as in the formulation of the problem. We note that since $(e_n)_{n=1}^\infty$ is an ON-basis for H we have

$$f_n = \Sigma_{k=1}^\infty \langle f_n, e_k\rangle e_k, \quad n = 1, 2, \ldots$$

and since T is continuous we get

$$Tf_n = \Sigma_{k=1}^\infty \langle f_n, e_k\rangle Te_k, \quad n = 1, 2, \ldots$$

This yields

$$\Sigma_{n=1}^\infty \|Tf_n\|^2 = \Sigma_{n=1}^\infty \Sigma_{k=1}^\infty \Sigma_{l=1}^\infty \langle f_n, e_k\rangle\langle Te_k, Te_l\rangle\overline{\langle f_n, e_l\rangle}.$$

and since the series is absolutely convergent we can change the order of summation. Observing that

$$\Sigma_{n=1}^\infty \langle f_n, e_k\rangle\overline{\langle f_n, e_l\rangle} = \langle \Sigma_{n=1}^\infty \langle e_l, f_n\rangle f_n, e_k\rangle = \langle e_l, e_k\rangle$$

we have

$$\Sigma_{n=1}^\infty \|Tf_n\|^2 = \Sigma_{k=1}^\infty \|Te_k\|^2.$$

Moreover $x = \Sigma_{n=1}^\infty \langle x, e_n\rangle e_n$ implies $Tx = \Sigma_{n=1}^\infty \langle x, e_n\rangle Te_n$ and

$$\|Tx\| = \|\Sigma_{n=1}^\infty \langle x, e_n\rangle Te_n\| \le \Sigma_{n=1}^\infty |\langle x, e_n\rangle|\|Te_n\| \le$$

$$\le (\Sigma_{n=1}^\infty |\langle x, e_n\rangle|^2)^{\frac{1}{2}}(\Sigma_{n=1}^\infty \|Te_n\|^2)^{\frac{1}{2}} = \|x\|(\Sigma_{n=1}^\infty \|Te_n\|^2)^{\frac{1}{2}}$$

by Parseval's formula. Hence we get

$$\|T\|^2 \le \Sigma_{n=1}^\infty \|Te_n\|^2.$$

Problem 3 Let
$$Mf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0$$
for $f \in L^2(\mathbb{R}_+)$. We observe that $Mf(x) \in \mathbb{R}$ for $x > 0$. Moreover for every continuous function f with compact support in $\{x \in \mathbb{R} : x > 0\}$ we note that
$$\int_0^\infty \frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2\,dt < \infty$$
and that
$$\|Mf\|^2 \leq \int_0^\infty \frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2\,dt = [-\frac{1}{t}(\int_0^t |f(s)|\,ds)^2]_0^\infty + 2\int_0^\infty \frac{1}{t}\int_0^t |f(s)|\,ds\,f(t)\,dt \leq$$
$$\leq 0 + 2(\int_0^\infty \frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2)^{\frac{1}{2}}(\int_0^\infty |f(s)|^2\,ds)^{\frac{1}{2}},$$
i.e. we get
$$\|Mf\| \leq 2\|f\|.$$
Now we recall that the set of continuous functions with compact support in $\{x \in \mathbb{R} : x > 0\}$ is dense in $L^2(\mathbb{R}_+)$ and from the inequality above we get that $Mf \in L^2$ for every $f \in L^2$ and that $\|M\| \leq 2$, and in particular that M is bounded. A straight-forward calculation show that
$$M^*f(x) = \int_x^\infty \frac{1}{t}f(t)\,dt, \quad x > 0$$
and that
$$\|(I - M)f\|^2 = \|f\|^2$$
for all $f \in L^2$. From this it follows that
$$\|I - M\| = 1.$$

Problem 4 & 5

See the textbook

Problem 6 We want to show that there exists a $C > 0$ such that for every $y \in \mathcal{R}(I + T)$ there exists a $x \in X$ with $(I + T)x = y$ satisfying
$$\|x\| \leq C\|y\|.$$
First we fix a $y \in \mathcal{R}(I + T)$ and set
$$d(y) = \inf\{\|x\| : (I + T)x = y\}.$$

Claim: There exists an $\tilde{x} \in X$ with $(I+T)\tilde{x} = y$ such that $\|\tilde{x}\| = d(y)$. To see this take a sequence $(x_n)_{n=1}^{\infty}$ with $(I+T)x_n = y$ such that $\|x_n\| \to d(y)$. Since this sequence is bounded there is a converging subsequence of $(Tx_n)_{n=1}^{\infty}$, let this still be denoted by $(Tx_n)_{n=1}^{\infty}$, converging to say $z \in X$. Here we used the fact that T is compact. But then $x_n \to y - z$ in X and hence $\tilde{x} = y - z$ has the desired property.

Now assume that there is no $C > 0$ with the property above. Then there are sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ satisfying $(I+T)x_n = y_n$ such that

$$\frac{\|\tilde{x}_n\|}{\|y_n\|} \to \infty.$$

Since T is linear we can without loss of generality assume that $\|\tilde{x}_n\| = 1$ for all n. Since T is compact there exists a converging subsequence of $(T\tilde{x}_n)_{n=1}^{\infty}$, call it still $(T\tilde{x}_n)_{n=1}^{\infty}$, converging to say v in X. We also have

$$\tilde{x}_n \to -v \quad \text{in} \quad X.$$

But since $\|y_n\| \to 0$ in X ($\|\tilde{x}_n\| = 1$ for all n) and since T is continuous (T is compact linear) we obtain $v = -Tv$. By the definition of $\tilde{x}_n$ this yields a contradiction since $(I+T)(\tilde{x}_n - v) = y_n$ and

$$\|\tilde{x}_n - v\| \geq \|\tilde{x}_n\| = 1$$

is valid for every n. The conclusion in problem 6 follows.

GLAD SOMMAR!!!

PS Omtentan äger rum den