Kortfattade lösningsskisser till tentamen i TMA401/MAN670, 2003-08-30

Problem 1

We will prove that

$$\begin{cases} u''(x) - u(x) = \lambda \frac{u(x)}{1+u^2(x)}, \quad x \in (0,1) \\ u(0) - u'(0) + u(1) = u(0) + u'(0) + 2u'(1) = 0 \\ u \in C^2 \end{cases}$$

has a unique solution for $|\lambda|$ small enough.

1. Determine the Green's function $g(x,t)$ for
   $$\begin{cases} u'' - u = F & x \in (0,1) \\ u(0) - u'(0) + u(1) = u(0) + u'(0) + 2u'(1) = 0 \end{cases}$$
   Set .... (standard calculation) ....
2. Set
   $$\begin{cases} (Tu)(x) = \int_0^1 g(x,t) \cdot \lambda \frac{u(t)}{1+u^2(t)}\, dt, \quad 0 \le x \le 1 \\ u \in C[0,1] \end{cases}$$
   The boundary value problem has a unique solution iff T has a unique fixed point. For $u, v \in C[0,1]$ we obtain
   $$|(Tu)(x) - (Tv)(x)| \le \text{......(standard calculations).....} \le C(\lambda)\|u - v\|_\infty.$$
   This shows that T is a contraction on the Banach space $C[0,1]$ provided $C(\lambda) < 1$ and the conclusion follows from Banach's fixed point theorem.

Problem 2

The solution is a straight forward application of the Gram-Schmidt process and we only refer to the textbook for more information.

Problem 3

We have to show that the ON-sequence $(u_n)_{n=1}^\infty$ in $L^2([0,1])$ is complete if

$$\sum_{n=1}^\infty \left| \int_0^x u_n(t)dt \right|^2 = x \text{ for all } x \in [0,1] \tag{1}$$

Formula (1) can be reformulated as

$$\sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n \rangle|^2 = \|\chi_{[0,x]}\|^2 \text{ for all } x \in [0,1],$$

where

$$\chi_I(t) = \begin{cases} 1 & t \in I \\ 0 & t \notin I \end{cases},$$

since

$$\langle \chi_{[0,x]}, u_n \rangle = \int_0^1 \chi_{[0,x]}(t)\overline{u_n(t)}dt = \overline{\int_0^x u_n(t)dt}$$

and

$$\|\chi_{[0,x]}\|^2 = \int_0^1 |\chi_{[0,x]}(t)|^2 dt = \int_0^x dt = x.$$

To show that $(u_n)_{n=1}^{\infty}$ is complete it is enough to show that

$$\sum_{n=1}^{\infty} |\langle f, u_n \rangle|^2 = \|f\|^2 \text{ for all } f \in L^2([0,1]). \tag{2}$$

Here it suffices to show that (2) is true for a dense set A in $L^2([0,1])$, since if $f_k \to f$ in L^2 and (2) is true for $f_k, k = 1, 2, \ldots$, then we have

$$\begin{aligned}
&0 \le \|f\|^2 - \sum_{n=1}^{\infty} |\langle f, u_n \rangle|^2 = \\
&= \|(f - f_k) + f_k\|^2 - \sum_{n=1}^{\infty} |\langle (f - f_k) + f_k, u_n \rangle|^2 \le \\
&\le \|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\| + \|f_k\|^2 - \\
&- \sum_{n=1}^{\infty} (\langle f - f_k, u_n \rangle + \langle f_k, u_n \rangle)\overline{(\langle f - f_k, u_n \rangle + \langle f_k, u_n \rangle)} = \\
&= \|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\|^2 + \sum_{n=1}^{\infty} |\langle f_k, u_n \rangle|^2 -
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n=1}^{\infty}(|\langle f-f_k,u_n\rangle|^2+2\mathrm{Re}\langle f-f_k,u_n\rangle\overline{\langle f_k,u_n\rangle}+\\
&+|\langle f_k,u_n\rangle|^2)\leq\\
&\leq\|f-f_k\|^2+2\|f_k\|\cdot\|f-f_k\|+\|f-f_k\|^2+\\
&+\sum_{n=1}^{\infty}|\langle f-f_k,u_n\rangle|\cdot|\langle f_k,u_n\rangle|\leq\\
&\leq 2\|f-f_k\|^2+2\|f_k\|\cdot\|f-f_k\|+\\
&+(\sum_{n=1}^{\infty}|\langle f-f_k,u_n\rangle|^2)^{1/2}(\sum_{n=1}^{\infty}|\langle f_k,u_n\rangle|^2)\leq\\
&\leq 2\|f-f_k\|^2+2\|f_k\|\cdot\|f-f_k\|+\|f-f_k\|\|f_k\|\to 0
\end{aligned}
$$

as $k\to\infty$. Here we have used that $\sup_k\|f_k\|<\infty$, which follows from $f_k\to f$ in L^2, and repeatedly applied Bessel's inequality.

Now we take A to be the set of all finite linear combinations of χ_I, where the I:s are subintervals of $[0,1]$. Then A is dense in $L^2([0,1])$. Set $g=\sum_{k=1}^N a_k\chi_{I_k}$ where the intervals I_k are pairwise disjoint subintervals of $[0,1]$ and $a_k\in\mathbb{C}$.

We know that (2) is valid for $f=\chi_{[0,x]}$. It remains to show that:

- $f=\chi_{[y,x]}$ satisfies (2)
- f,g satisfies (2) and have disjoint support implies that any linear combination of f,g satisfies (2).

Fix $0<y<x\leq 1$. We note that

$$\|\chi_{[0,x]}\|^2=\sum_{n=1}^{\infty}|\langle\chi_{[0,x]},u_n\rangle|^2$$

$$\|\chi_{[0,y]}\|^2=\sum_{n=1}^{\infty}|\langle\chi_{[0,y]},u_n\rangle|^2. \tag{3}$$

This yields

$$\|\chi_{[y,x]}\|^2=x-y$$

and

$$
\begin{aligned}
&\sum_{n=1}^{\infty} |\langle \chi_{[y,x]}, u_n \rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]} - \chi_{[0,y]}, u_n \rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n \rangle - \langle \chi_{[0,y]}, u_n \rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle \chi_{[0,y]}, u_n \rangle|^2 - \\
&- 2\mathrm{Re} \sum_{n=1}^{\infty} \langle \chi_{[0,x]}, u_n \rangle \overline{\langle \chi_{[0,y]}, u_n \rangle} = \\
&= x + y - 2\mathrm{Re}\langle \chi_{[0,x]}, \sum_{n=1}^{\infty} \langle \chi_{[0,y]}, u_n \rangle u_n \rangle = (!) = \\
&= x + y - 2\mathrm{Re}\langle \chi_{[0,x]}, \chi_{[0,y]} \rangle = x + y - 2\|\chi_{[0,y]}\|^2 = x - y.
\end{aligned}
$$

Note that at (!) we have used the fact

$$
\chi_{[0,y]} = \sum_{n=1}^{\infty} \langle \chi_{[0,y]}, u_n \rangle u_n
$$

which follows from (3).

We have thus found that $\chi_{[y,x]}$ satisfies (2). To prove the second statement we have to do similar calculations (do it yourself!!!) and the full statement that $g \in A$ implies g satisfies (2) follows by induction over N (see the expression for g above).

Problem 4 & 5 & 6

See the textbook