Matematik, CTH & GU
Tentamen i Funktionalanalys TMA401/MAN670
Hjälpmedel: Inga (inte ens räknedosa).
Personuppgifter: Namn, personnummer, linje, antagningsår.
Inlämning ska ske i uppgifternas ordning; v.g. sidnumrera!
Teoriuppgifter: 4,5,6.
Telefon: Axel Målqvist 0740-459022

Datum: 2003–05–31
Skrivtid: fm (5 timmar)

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1. Prove the existence and uniqueness of a solution to the following boundary value problem

$$\begin{cases} u''(x) + u'(x) = \arctan u(x^2), \quad 0 \le x \le 1 \\ u(0) = u(1) = 0, \quad u \in C^2([0,1]) \end{cases}$$

(4p)

2. Let $(e_n)_{n=1}^{\infty}$ be an ON-basis for a Hilbert space H and assume that $T : H \to H$ is a bounded linear operator on H such that

$$\Sigma_{n=1}^{\infty} \|Te_n\|^2 < \infty.$$

Show that if $(f_n)_{n=1}^{\infty}$ is another ON-basis for H then

$$\Sigma_{n=1}^{\infty} \|Tf_n\|^2 = \Sigma_{n=1}^{\infty} \|Te_n\|^2.$$

Moreover show that

$$\|T\|^2 \le \Sigma_{n=1}^{\infty} \|Te_n\|^2.$$

(4p)

3. Set $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$. For $f \in L^2(\mathbb{R}_+)$ define

$$Mf(x) = \frac{1}{x} \int_0^x f(t)\, dt, \quad x > 0.$$

Show that

$$M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

is a bounded linear mapping on $L^2(\mathbb{R}_+)$, calculate the operator norm of $I - M$ and, finally, determine the adjoint operator of M. Here I denotes the identity operator on $L^2(\mathbb{R}_+)$.

(4p)

4. State and prove the Riesz representation theorem.

(5p)

5. Let $\mathcal{K}(H)$ be the subset of all compact linear operators $H \to H$ on a Hilbert space H in $\mathcal{B}(H)$ with the operator norm. Show that $\mathcal{K}(H)$ is a closed subspace in $\mathcal{B}(H)$.

(4p)

6. Let X be a Banach space and $T : X \to X$ a compact[1] linear operator. Show that there exists a constant C such that for every $y \in \mathcal{R}(I + T)$ there exists a $x \in X$ with $y = (I + T)x$ such that

$$\|x\| \leq C\|y\|.$$

(4p)

Good Luck!!
PK

[1] Exactly the same definition as for a linear operator on a Hilbert space