



14th May 2004

## TMA401 Functional Analysis

### MAN670 Applied Functional Analysis

### 4th quarter 2003/2004

All document concerning the course can be found on the course home page:  
<http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/>

## Home Assignment 2

**Problem 1:** Let  $T : H \rightarrow H$  be a compact linear operator on a Hilbert space  $H$ . Show that  $I + T$  is compact if and only if  $H$  is finite-dimensional. Here  $I$  denotes the identity operator on  $H$ .

**Problem 2:** Set

$$Tf(x) = \int_0^\pi \cos(x-y)f(y) dy, \quad 0 \leq x \leq \pi.$$

Find the norm of  $T$  where  $T$  is regarded as an operator on  $L^2([0, \pi])$ .

**Problem 3:** Prove the existence and uniqueness of solution to the following boundary value problem:

$$\begin{cases} 4u''(x) = |x + u(x)|, & 0 \leq x \leq 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, & u \in C^2([0, 1]). \end{cases}$$

**Problem 4:** Let  $(x_n)_{n=1}^\infty$  be a bounded sequence in a separable Hilbert space  $H$ . Show that there exists a subsequence  $(x_{n_k})_{k=1}^\infty$  and an  $x \in H$  such that

$$x_{n_k} \xrightarrow{w} x.$$

What happens if  $H$  is not separable?

**Problem 5:** Let  $T : H \rightarrow H$  be a compact positive self-adjoint operator on a Hilbert space  $H$ . Moreover assume that  $\|T\| \leq 2$ . Give an estimate<sup>1</sup> for

$$\|T^2 - 3T + I\|.$$

The solutions should be handed in at the latest on **Friday May 21**.

<sup>1</sup>better than the trivial estimate  $\|T^2 - 3T + I\| \leq 11$ .