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Matematiska Institutionen
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TMA401 Functional Analysis
MAN670 Applied Functional Analysis
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All document concerning the course can be found on the course home page:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/

Solutions to home assignments (sketches)

Problem 1: Let Y be a finite-dimensional subspace of a normed space X. Show that Y is closed.

Solution: It is enough to show that if $(y_n)_{n=1}^{\infty}$ is a sequence in Y converging to some element y in X then $y \in Y$. So assume that $(y_n)_{n=1}^{\infty}$ is a sequence in Y convering in X and call the limit element y. Fix a basis $e_1, e_2, \ldots, e_n$ in Y. Every element y_n can be written in the form

$$y_n = \Sigma_{k=1}^{n} \alpha_k^{(n)} e_k.$$

Moreover all norms on finite-dimensional spaces, here we consider the space Y with the induced norm from X, are equivalent and we see that $\|z\| = \Sigma_{k=1}^{n} |\alpha_k|$, where $z = \Sigma_{k=1}^{n} \alpha_k e_k$, defines a norm. This implies that $(\alpha_k^{(n)})_{n=1}^{\infty}$, $k = 1, 2, \ldots, n$, are Cauchy sequences in $\mathbb{C}$ (if Y is a complex normed space) and hence converges. Call the limits $\tilde{\alpha}_k$, $k = 1, 2, \ldots, n$. Set $\tilde{y} = \Sigma_{k=1}^{n} \tilde{\alpha}_k e_k$. This implies that $y_n \to \tilde{y}$ in Y and since $y_n \to y$ in X we have $y = \tilde{y} \in Y$. This proves that Y is closed.

Problem 2: Show that l^1 (as a vector space) is a subspace of l^2. Is this subspace closed in l^2 with the l^2-norm?

Solution: Let $\mathbb{x} = (x_1, x_2, \ldots) \in l^1$. Since

$$\Sigma_{k=1}^{n} |x_k|^2 \leq (\Sigma_{k=1}^{n} |x_k|)^2$$

holds true for every positive integer n we obtain

$$\|\mathbb{x}\|_{l^2} \leq \|\mathbb{x}\|_{l^1} < \infty$$

and $\mathbb{x} \in l^2$. Moreover since l^1 is a vector space we see that l^1 is a subspace of l^2. To see that l^1 is not closed in l^2 consider for instance the sequence $(\mathbb{x}_n)_{n=1}^{\infty}$ where

$$\mathbb{x}_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots).$$

Here $\mathbb{x}_n \in l^1$ for all n and $\mathbb{x}_n \to \mathbb{x}$ in l^2 where $\mathbb{x} = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$. This is clear since

$$\|\mathbb{x}_n - \mathbb{x}\|_{l^2} = (\Sigma_{k=n+1}^{\infty} |\frac{1}{k}|^2)^{\frac{1}{2}} \to 0$$

as $n \to \infty$.

Problem 3: Let X be a normed space. Show that X is finitedimensional if and only if every closed and bounded set in X is compact.

Solution: Will be given later since it is quite long, but not very difficult, to prove using Riesz lemma in ”one direction”.

Problem 4: Set $X = l^2$ with the $\| \|_{l^2}$-norm and define the mappings T_1, T_2 by

$$T_1(x_1, x_2, x_3, \ldots, x_n, \ldots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots, \frac{1}{n}x_n, \ldots)$$

and

$$T_2(x_1, x_2, x_3, \ldots, x_n, \ldots) = (x_1, x_2^2, x_3^3, \ldots, x_n^n, \ldots)$$

for $(x_1, x_2, x_3, \ldots, x_n, \ldots) \in l^2$. Is T_1 a linear mapping? Is T_2 a linear mapping? Is T_1 continuous at any point in l^2? Is T_2 continuous at any point in l^2? Calculate

$$\sup\{\|T(x_1, x_2, x_3, \ldots, x_n, \ldots)\|_{l^2} : \|(x_1, x_2, x_3, \ldots, x_n, \ldots)\|_{l^2} \leq r\}$$

for all $r > 0$ for both T equal to T_1 and to T_2. Explain the difference.

Solution: It is easily seen that T_1 is a linear mapping and that T_2 is not a linear mapping on l^2. It is also easy to see that the operator norm of T_1 is equal to 1 and that

$$\sup\{\|T_1(\mathbb{x})\| : \|\mathbb{x}\| < r\} = r$$

for every $r > 0$. Since T_1 is a bounded linear mapping it is also continuous at every $\mathbb{x} \in l^2$.

It remains to treat the mapping T_2. First we see that $T_2(\mathbb{x}) \in l^2$ for every $\mathbb{x} \in l^2$. To see this we fix a $\mathbb{x} = (x_1, x_2, x_3, \ldots) \in l^2$. From $(\Sigma_{n=1}^{\infty}|x_n|^2)^{1/2} < \infty$ it follows that there exists an integer N such that $|x_n| \leq 1$ for all $n \geq N$. This implies that

$$\begin{aligned} &\|T_2(x_1, x_2, x_3, \ldots)\|_{l^2} = \|(x_1, x_2^2, x_3^3, \ldots)\|_{l^2} \leq \\ &\leq \{\Delta - \text{inequality}\} \leq \\ &\leq \|(x_1, x_2^2, \ldots, x_{N-1}^{N-1}, 0, 0, \ldots)\|_{l^2} + \|(0, 0, \ldots, 0, x_N^N, x_{N+1}^{N+1}, \ldots)\|_{l^2} \leq \\ &\leq \underbrace{\|(x_1, x_2^2, \ldots, x_{N-1}^{N-1}, 0, 0, \ldots)\|_{l^2}}_{<\infty} + \underbrace{\|(0, 0, \ldots, 0, x_N, x_{N+1}, \ldots)\|_{l^2}}_{<\infty} < \infty \end{aligned}$$

and hence $\mathbb{x} \in l^2$.

We easily see that

$$\sup\{\|T_2(\mathbb{x})\| : \|\mathbb{x}\| < r\} = \begin{cases} r & 0 \leq r \leq 1 \\ \infty & 1 < r. \end{cases}$$

Here the last statement follows e.g. from the observation that

$$T_2(0, 0, \ldots, \underbrace{\frac{r+1}{2}}_{\text{position } n}, 0, \ldots) = (0, 0, \ldots, \underbrace{(\frac{r+1}{2})^n}_{\text{position } n}, 0, \ldots)$$

and letting $n \to \infty$.

Finally we observe that T_2 is continuous at every $\mathbb{x} \in l^2$. To prove this fix a $\mathbb{x} \in l^2$ and a sequence $(\mathbb{x}_k)_{k=1}^{\infty}$ in l^2 such that $\mathbb{x}_k \to \mathbb{x}$ in l^2. Set

$$\mathbb{x} = (x_1, x_2, x_3, \ldots)$$

and

$$\mathbb{x}_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots),\ k = 1, 2, \ldots.$$

Since $\mathbb{x} \in l^2$ there exists an integer N such that

$$|x_n| < \frac{1}{2} \quad \text{for all} \quad n \geq N,$$

and since $\mathbb{x}_k \to \mathbb{x}$ i l^2 there exists an integer K such that

$$\|\mathbb{x}_k - \mathbb{x}\|_{l^2} < \frac{1}{4} \quad k \geq K.$$

This implies that

$$|x_n^{(k)}| \leq |x_n^{(k)} - x_n| + |x_n| < \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

for all $n \geq N$, $k \geq K$. Now fix an $\epsilon > 0$. We see that

$$\begin{aligned} &\|T_2(\mathbb{x}_k) - T_2(\mathbb{x})\|_{l^2} = (\Sigma_{n=1}^{\infty}|(x_n^{(k)})^n - (x_n)^n|^2)^{1/2} \leq \\ &\leq \{\Delta - \text{inequality}\} \leq \\ &\leq (\Sigma_{n=1}^{N-1}|(x_n^{(k)})^n - (x_n)^n|^2)^{1/2} + (\Sigma_{n=N}^{\infty}|(x_n^{(k)})^n - (x_n)^n|^2)^{1/2} \end{aligned}$$

where

$$\begin{aligned} &|(x_n^{(k)})^n - (x_n)^n| \leq |x_n^{(k)} - x_n|\Sigma_{l=0}^{n-1}|x_n^{(k)}|^l|x_n|^{n-1-l} \leq \\ &\leq |x_n^{(k)} - x_n| \cdot n(\frac{3}{4})^{n-1} \end{aligned}$$

for all $n \geq N$ and $k \geq K$. However $\sup_{n\in\mathbb{Z}_+} n \cdot (\frac{3}{4})^{n-1} \equiv C < \infty$. This finally implies that

$$\|T_2(\mathbb{x}_k) - T_2(\mathbb{x})\|_{l^2} \leq \underbrace{(\Sigma_{n=1}^{N-1}|(x_n^{(k)})^n - (x_n)^n|^2)^{1/2}}_{\substack{\to 0 \\ \text{since } \mathbb{x}_k \to \mathbb{x} \text{ in } l^2 \\ \text{because it implies that} \\ x_n^{(k)} \to x_n, k\to\infty \\ \text{for all } n.}} + C\underbrace{(\Sigma_{n=N}^{\infty}|x_n^{(k)} - x_n|^2)^{1/2}}_{\substack{\to 0 \\ \text{since } \mathbb{x}_k \to \mathbb{x} \text{ in } l^2}}$$

Note that we have proven that both T_1 and T_2 are continuous at every point. However, continuity for a nonlinear mapping does not imply that it maps bounded sets onto bounded sets while this is true for every linear mapping.

Problem 5: Let X be a Banach space and let $T_n \in \mathcal{B}(X, X)$, $n = 1, 2, 3, \ldots$ Assume that $\lim_{n\to\infty} T_n x$ exists for every $x \in X$. Show that $T \in \mathcal{B}(X, X)$ where T is defined by

$$Tx = \lim_{n\to\infty} T_n x$$

for $x \in X$.

Solution: Clearly T is a linear mapping on X since

$$\begin{aligned} &T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) = \lim_{n\to\infty} (\alpha T_n(x) + \beta T_n(y)) = \\ &= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y) = \alpha T(x) + \beta T(y) \end{aligned}$$

for every $x, y \in X$ and all scalars α, β. Moreover since for all $x \in X$ the sequence $(T_n(x))_{n=1}^{\infty}$ converges, and hence is bounded, we conclude from the Banach-Steinhaus Theorem that the sequence $(\|T_n\|)_{n=1}^{\infty}$ is bounded. This yields

$$\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\| \leq (\sup_n \|T_n\|)\|x\|,$$

for all $x \in X$. This shows that T is a bounded linear mapping on X.

Problem 6: Let $T : H \to H$ be a compact linear operator on a Hilbert space H. Show that $I + T$ is compact if and only if H is finite-dimensional. Here I denotes the identity operator on H.

Solution: If H is an infinite-dimensional Hilbert space there exists an ON-sequence $(e_n)_{n=1}^{\infty}$ in H. Here $e_n \rightharpoonup 0$ and so $T(e_n) \to 0$ since T is compact. From this we see that the sequence $((I + T)(e_n))_{n=1}^{\infty}$ can not have any convergent subsequence since for $n \neq m$

$$\sqrt{2} = \|e_n - e_m\| \leq \|(I + T)(e_n) - (I + T)(e_m)\| + \|T(e_n)\| + \|T(e_m)\|$$

and so

$$\|(I + T)(e_n) - (I + T)(e_m)\| \geq \sqrt{2} - \|T(e_n)\| - \|T(e_m)\| \to \sqrt{2}$$

as $n, m \to \infty, n \neq m$. On the other hand, if H is finite-dimensional Hilbert space then I is a compact operator and so $I + T$ is compact.

Problem 7: Set

$$Tf(x) = \int_0^{\pi} \cos(x - y)f(y)\,dy, \quad 0 \leq x \leq \pi.$$

Find the norm of T where T is regarded as an operator on $L^2([0, \pi])$.

Solution: (sketch) T is a self-adjoint ($k(x,y) = \cos(x-y)$ satisfies $k(x,y) = \overline{k(y,x)}$) compact ($k \in L^2([0,\pi]\times[0,\pi])$) linear operator on the Hilbert space $L^2([0,\pi])$. Hence $\|T\| = \sup_{\lambda \text{ eigenvalue}} |\lambda|$. It is an easy exercise to calculate the eigenvalues to T.

Problem 8: Prove the existence and uniqueness of solution to the following boundary value problem:

$$\begin{cases} 4u''(x) = |x+u(x)|, \quad 0 \le x \le 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, \quad u \in C^2([0,1]). \end{cases}$$

Solution: Standard problem. Calculations are omitted.

Problem 9: Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in a separable Hilbert space H. Show that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ and an $x \in H$ such that

$$x_{n_k} \rightharpoonup x.$$

What happens if H is not separable?

Solution: (sketch) Assume that H is a separable Hilbert space and that $(e_k)_{k=1}^{\infty}$ is an ON-basis. Applying a "diagonal sequence"-argument as in Theorem 4.8.5 we obtain a subsequence $(x_{p_n})_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\langle x_{p_n}, e_k\rangle$ converges for all $k = 1, 2, \dots$ Call the limits α_k. Set $M = \sup_n \|x_n\|$. Here $M < \infty$ by the hypothesis. Note that

$$\Sigma_{k=1}^{\infty} |\langle x_{p_n}, e_k\rangle|^2 = \|x_{p_n}\| \le M^2$$

by Parseval's formula and letting $n \to \infty$ we conclude

$$\Sigma_{k=1}^{\infty} |\alpha_k|^2 \le M^2.$$

Now fix an arbitrary $x \in H$. We obtain

$$\begin{aligned} &\langle x_{p_n}, x\rangle = \langle x_{p_n}, \Sigma_{k=1}^{\infty}\langle x, e_k\rangle e_k\rangle = \\ &= \Sigma_{k=1}^{\infty}\overline{\langle x, e_k\rangle}\langle x_{p_n}, e_k\rangle = \Sigma_{k=1}^{\infty}\overline{\langle x, e_k\rangle}\alpha_k + \Sigma_{k=1}^{\infty}\overline{\langle x, e_k\rangle}(\langle x_{p_n}, e_k\rangle - \alpha_k) \end{aligned}$$

and so

$$|\langle x_{p_n}, x\rangle - \Sigma_{k=1}^{\infty}\overline{\langle x, e_k\rangle}\alpha_k| \le \Sigma_{k=1}^{N}|\overline{\langle x, e_k\rangle}||(\langle x_{p_n}, e_k\rangle - \alpha_k)| + \Sigma_{k=N+1}^{\infty}|\overline{\langle x, e_k\rangle}||(\langle x_{p_n}, e_k\rangle - \alpha_k)|.$$

For fixt N the first term on the RHS tends to 0 as $n \to \infty$ while the second term can be estimated from above, using the Cauchy-Schwartz inequality, by

$$(\Sigma_{k=N+1}^{\infty}|\langle x, e_k\rangle|^2)^{\frac{1}{2}} 2M.$$

which tends to 0 as $N \to \infty$.Hence we have that $|\langle x_{p_n}, x\rangle$ converges as $n \to \infty$. (We also see that $x_{p_n} \to \Sigma_{k=1}^{\infty}\alpha_k e_k$)

Finally, if H is not separable consider let $\tilde{H}$ denote the closure of the linear span of the set $\{x_n : n = 1, 2, \dots\}$. Then $\tilde{H}$ is a Hilbert space containing all x_n. Moreover $\tilde{H}$ is separable (possibly finite-dimensional) since an ON-basis can be constructed by the Gram-Schmidt process applied to the sequence $(x_n)_{n=1}^{\infty}$. By the construction above we have a subsequence $(x_{p_n})_{n=1}^{\infty}$ that converges weakly on $\tilde{H}$. Now $\langle x_{p_n}, x\rangle$ converges for every $x \in H$ as $n \to \infty$ since every x can be decomposed as $y + z$, $y \in \tilde{H}$ and $z \in \tilde{H}^{\perp}$ and $\langle x_{p_n}, z\rangle = 0$.

Problem 10: Let $T : H \to H$ be a compact positive self-adjoint operator on a Hilbert space H. Moreover assume that $\|T\| \le 2$. Give an estimate[1] for

$$\|T^2 - 3T + I\|.$$

[1] better than the trivial estimate $\|T^2 - 3T + I\| \le 11$.

Solution: (sketch) Applying the Hilbert-Schmidt theorem we have an ON-sequence $(e_n)_{n=1}^{\infty}$ of eigenvectors corresponding to the non-zero eigenvalues $(\lambda_n)_{n=1}^{\infty}$ to T such that $T|_S = 0$, where $S = \overline{\text{Span}\{e_n : n = 1, 2, \ldots\}}^{\perp}$. Moreover we know that $0 < \lambda_n \leq 2$ for all n. This yields

$$\|(T^2-3T+I)x\|^2 = \|\Sigma_{n=1}^{\infty}(\lambda_n^2-3\lambda_n+1)\langle x, e_n\rangle e_n\|^2 = \Sigma_{n=1}^{\infty}|\lambda_n^2-3\lambda_n+1|^2|\langle x, e_n\rangle|^2 \leq (\frac{5}{4}\|x\|)^2$$

for all $x \in \overline{\text{Span}\{e_n : n = 1, 2, \ldots\}}$. Here we have used Parseval's formula together with

$$\max_{0\leq x\leq 2} |x^2 - 3x + 1| = \frac{5}{4}.$$

For $z \in S$ we get $(T^2 - 3T + I)(z) = z$ and hence $\|(T^2 - 3T + I)(z)\| = \|z\|$. Finally if $x \in H$ then $x = y + z$, where $y \in \overline{\text{Span}\{e_n : n = 1, 2, \ldots\}}$ and $z \in S = \overline{\text{Span}\{e_n : n = 1, 2, \ldots\}}^{\perp}$, we get

$$\begin{aligned}
&\|(T^2 - 3T + I)(x)\|^2 = \|(T^2 - 3T + I)(y + z)\|^2 = \\
&= \|(T^2 - 3T + I)(y)\|^2 + \|(T^2 - 3T + I)(z)\|^2 \leq \\
&\leq (\frac{5}{4})^2\|y\|^2 + \|z\|^2 \leq (\frac{5}{4})^2(\|y\|^2 + \|z\|^2) = (\frac{5}{4})^2\|x\|^2.
\end{aligned}$$

We conclude that $\|T^2 - 3T + I\| \leq \frac{5}{4}$.

The written exam will take place in the V-building on May 29.