CHALMERS | GÖTEBORGS UNIVERSITET
Matematiska Institutionen
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8th May 2005

TMA401 Functional Analysis
MAN670 Applied Functional Analysis
4th quarter 2004/2005

All document concerning the course can be found on the course home page:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/

Home Assignment 2

Problem 1: Let $x_1, x_2, \dots, x_N$ be linearly independent vectors in an inner product space, with $N = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. Show that there are orthonormal vectors $y_1, y_2, \dots, y_n$ such that

$$y_i = \Sigma_{j \in A_i} \lambda_j x_j, \; i = 1, 2, \dots, n,$$

where $A_1, A_2, \dots, A_n$ are disjoint subsets of $\{1, 2, \dots, N\}$.

Problem 2: Let X be a separable Hilbert space. Show that the closure in $\mathcal{B}(X, X)$ of the vector space of all finite dimensional operators is equal to $\mathcal{K}(X, X)$.

Problem 3: Let a_n, $n = 1, 2, 3, \dots$ be non-negative reals and set

$$C = \{x \in l^2 : x = (x_n)_{n=1}^{\infty}, \, |x_n| \le a_n \text{ all } n\}.$$

Show that if C is a compact subset in l^2 then $a_n \to 0$ as $n \to \infty$. For what sequences $(a_n)_{n=1}^{\infty}$ is C compact?

Problem 4: Let T be defined on $L^2([0,1])$ by $Tf(x) = \int_0^x f(y)\, dy$. Show that T is a compact operator on $L^2([0,1])$ with $\sigma(T) = \{0\}$. In particular prove that T has no eigenvalues $\neq 0$.

Problem 5: Let $f(x)$ be a complex-valued function on $\mathbb{R}_+ = \{x : x > 0\}$ and let Lf be the function defined on $\mathbb{R}_+$ by

$$Lf(x) = \int_0^{\infty} f(y) e^{-xy}\, dy.$$

Show that L is a bounded[1] linear mapping $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ with $\|L\| \le \sqrt{\pi}$. Show that L is not a bounded[2] linear mapping $L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ for $p \neq 2$.

Problem 6: Prove the existence and uniqueness of a solution to the following boundary value problem:

$$\begin{cases} 4u''(x) = |x + u(x)|, & 0 \le x \le 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, & u \in C^2([0,1]). \end{cases}$$

The solutions should be handed in at the latest on **Friday May 20**.

[1] Hint: write $f(y)e^{-yx} = (f(y)e^{-yx/2}y^{1/4})(e^{-yx/2}y^{-1/4})$ and use Hölder's inequality.
[2] Hint: Try $f(x) = e^{-ax}$