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1 A Note on Ordinary Differential Equations

1.1 Introduction

Let $c_0, \ldots, c_n \in C(I)$ be fixed, where $I = [a, b]$, $n \geq 2$ and

$$c_n(x) \neq 0, \text{ for all } x \in I.$$

Set

$$Lu = c_n u^{(n)} + \ldots + c_0 u, \; u \in C^n(I).$$

The aim of this note is to show that the differential operator L with proper homogeneous boundary conditions has a so called Green's function. This means that solution can be written as an integral with the Green's function appearing as the kernel function. Moreover we show that provided the operator L is symmetric the solution has a spectral decomposition. This follows from the spectral theorem for compact self-adjoint operators on Hilbert spaces ([1] Theorem 4.10.2).

1.2 Existence of Green's functions

Our first result is the following fundamental existence theorem for ordinary differential equations.

Theorem 1.1. *Assume $t_0 \in I$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. Then for every $f \in C(I)$ there exists a unique $u \in C^n(I)$ such that $Lu = f$ and $(u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) = \xi$.*

Proof. Set $y_1 = u$, $y_2 = u', \ldots, y_n = u^{(n-1)}$. The equation $Lu = f$ is equivalent to

$$\begin{cases} y_1' = y_2 \\ \vdots \\ y_{n-1}' = y_n \\ \\ y_n' = -\frac{c_0}{c_n} y_1 - \ldots - \frac{c_{n-1}}{c_n} y_n + \frac{1}{c_n} f \end{cases}$$

or, using the vector notation $y = (y_1, \ldots, y_n)$,

$$y' = F(t, y), \; t \in I$$

for a vector-valued function F. This function satisfies a so called Lipschitz condition

$$|F(t,y) - F(t,z)| \leq K|y-z|,\ t \in I,\ y,z \in \mathbb{R}^n,$$

for some $K \in \mathbb{R}$. Moreover note that the condition $(u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) = \xi$ can be written $y(t_0) = \xi$. Picard's existence theorem ([1] theorem 5.2.5) in vector form yields the result. □

We introduce the notation

$$\mathcal{N}(L) = \{u \in C^n(I); Lu = 0\}.$$

Clearly $\mathcal{N}(L)$ is a subspace of $C^n(I)$ since L is a linear operator.

Corollary 1.1. $\dim \mathcal{N}(L) = n$.

Proof. Let $t_0 \in I$ be fixed and define

$$Tu = (u(t_0), \ldots, u^{(n-1)}(t_0)),\ u \in \mathcal{N}(L).$$

The linear mapping $T : \mathcal{N}(L) \to \mathbb{C}^n$ is a bijection from the previous theorem with the range $\mathbb{C}^n$. Hence we get $\dim \mathcal{N}(L) = \dim \mathbb{C}^n = n$. □

For arbitrary functions $u_1, \ldots, u_n \in \mathcal{N}(L)$ we define the **Wronskian** for $u_1, \ldots, u_n$ by

$$W(t) = \begin{vmatrix} u_1(t) & u_2(t) & \ldots & u_n(t) \\ u_1'(t) & u_2'(t) & & u_n'(t) \\ \vdots & \vdots & & \vdots \\ u_1^{(n-1)}(t) & u_2^{(n-1)}(t) & & u_n^{(n-1)}(t) \end{vmatrix},\ t \in I.$$

Theorem 1.2. *The following conditions are equivalent:*

1. $W(t) \neq 0$ *for all* $t \in I$.
2. $W(t_0) \neq 0$ *for some* $t_0 \in I$.
3. $u_1, \ldots, u_n$ *is a basis for the vector space* $\mathcal{N}(L)$.

Proof. (1) $\Rightarrow$ (2): trivial.

(2) $\Rightarrow$ (3): Take an $u \in \mathcal{N}(L)$. Since $\dim \mathcal{N}(L) = n$ it is enough to show that u is a linear combination of $u_1, \ldots, u_n$.

Assume that $t_0 \in I$ is fixed and that $W(t_0) \neq 0$. Let $u \in \mathcal{N}(L)$. From courses in linear algebra we know that there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^n$ such that

$$\sum_{k=1}^{n} \alpha_k (u_k(t_0), \ldots, u_k^{(n-1)}(t_0)) = (u(t_0), \ldots, u^{(n-1)}(t_0)).$$

The function $v = \sum_1^n \alpha_k u_k \in \mathcal{N}(L)$ satisfies the relation

$$(v(t_0), \dots, v^{(n-1)}(t_0)) = (u(t_0), \dots, u^{(n-1)}(t_0))$$

and by Theorem 1.1 we have $v = u$. Hence it follows that $u \in \text{span}\,\{u_1, \dots, u_n\}$.

(3) $\Rightarrow$ (1): Let $t \in I$ be arbitrary. We will show that $W(t) \neq 0$. It is enough to show that the columns in the determinant $W(t)$ are linearly independent.

Assume that $\alpha_1, \dots, \alpha_n \in \mathbb{C}^n$ and that

$$\sum_{k=1}^{n} \alpha_k (u_k(t), \dots, u_k^{(n-1)}(t)) = 0.$$

The function $v = \sum_1^n \alpha_k u_k \in \mathcal{N}(L)$ satisfies $v(t) = \dots = v^{(n-1)}(t) = 0$ and is equal to the zero function by Theorem 1.1. However from $\sum_1^n \alpha_k u_k = \mathbf{0}$ it follows that $\alpha_1 = \dots = \alpha_n = 0$. Hence the columns in the determinant $W(t)$ are linearly independent. $\square$

From now on we use the following notation:

$$\alpha_{ij}, \beta_{ij}, i = 0, \dots, n-1,\ j = 1, \dots, n$$

are complex numbers and

$$R_j u = \sum_{i=0}^{n-1} [\alpha_{ij} u^{(i)}(a) + \beta_{ij} u^{(i)}(b)],\ j = 1, \dots, n.$$

are boundary operators. Moreover we set

$$Ru = (R_1 u, \dots, R_n u)$$

$$C_R^n(I) = \{u \in C^n(I) : Ru = 0\}$$

and

$$L_0 u = Lu,\ u \in C_R^n(I).$$

Theorem 1.3. *The following conditions are equivalent:*

1. *The mapping $L_0 : C_R^n(I) \to C(I)$ is a bijection.*
2. $\det\{R_j u_k\}_{1 \le j,k \le n} \neq 0$ *for every (alternatively for some) basis $u_1, \dots, u_n$ i $\mathcal{N}(L)$.*

Proof. (1) $\Rightarrow$ (2): If the determinant in (2) is zero then there are $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ not all equal to zero such that

$$\sum_{k=1}^{n} \alpha_k R_j u_k = 0,\ j = 1, \dots, n.$$

The function $v = \sum_1^n \alpha_k u_k$ satisfies $Lv = 0$ together with $Rv = 0$. This yields a contradiction since $v \neq 0$ and $L_0 v = 0$.

(2) $\Rightarrow$ (1): Take an arbitrary $f \in C(I)$. It remains to prove that the equation

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases}$$

is uniquely solvable. Set $w = u - v$, where $v \in C^n(I)$ satisfies $Lv = f$ (Theorem 1.1), we obtain the equivalent equation

$$\begin{cases} Lw = 0 \\ Rw = -Rv. \end{cases}$$

With the ansatz $w = \sum_1^n \alpha_k u_k$ the determinant condition in (2) gives the existence of a unique solution. $\square$

Now let $u_1, \dots, u_n$ be a basis for the vector space $\mathcal{N}(L)$ and set

$$e(x,t) = \sum_{k=1}^{n} a_k(t) u_k(x)$$

where $a_1(t), \dots, a_n(t)$ are chosen such that

$$\begin{cases} e_x^{(k)}(t,t) = 0,\ k = 0, 1, \dots, n-2 \\ e_x^{(n-1)}(t,t) = 1/c_n(t). \end{cases}$$

Note that the functions $a_1(t), \dots, a_n(t)$ are continuous in t due to Cramer's rule. Also observe that for fixed $t \in I$ the function $u(x) = e(x,t)$ is the unique solution to the equation

$$\begin{cases} Lu = 0 \\ u(t) = \dots = u^{(n-2)}(t) = 0,\ u^{(n-1)}(t) = 1/c_n(t). \end{cases}$$

The function $e(x,t), (x,t) \in I \times I$, is called the **fundamental solution** to the operator L. This function is of interest in connection with boundary value problems that we will discuss next.

Theorem 1.4. *Let $u_1, \dots, u_n$ be a basis for $\mathcal{N}(L)$ such that*

$$\det\{R_j u_k\}_{1 \leq j,k \leq n} \neq 0$$

and set $G = L_0^{-1}$. Then there exists a unique continuous function $g(x,t), (x,t) \in I \times I$, such that

$$(Gf)(x) = \int_I g(x,t) f(t) dt.$$

This is called the Green's function g and can be constructed as follows:

1. *Set $\tilde{e}(x,t) = \theta(x-t) e(x,t)$, where θ is the Heaviside's function*

2. Determine $b_1, \ldots, b_n \in C(I)$ *such that the function*

$$g(x,t) = \tilde{e}(x,t) + \sum_{k=1}^{n} b_k(t) u_k(x)$$

satisfies

$$R(g(\cdot, t)) = 0,\ a < t < b.$$

Proof. First set

$$\tilde{u}(x) = \int_I \tilde{e}(x,t) f(t) dt,$$

i.e.

$$\tilde{u}(x) = \int_a^x e(x,t) f(t) dt.$$

Repeated differentiations yield

$$\begin{aligned}
\tilde{u}'(x) &= \int_a^x e'_x(x,t) f(t) dt + \underbrace{e(x,x)}_{=0} f(x) \\
\tilde{u}''(x) &= \int_a^x e''_x(x,t) f(t) dt + \underbrace{e'_x(x,x)}_{=0} f(x) \\
&\vdots \\
\tilde{u}^{(n-1)}(x) &= \int_a^x e_x^{(n-1)}(x,t) f(t) dt + \underbrace{e^{(n-2)}(x,x)}_{=0} f(x)
\end{aligned}$$

and

$$\tilde{u}^{(n)}(x) = \int_a^x e^{(n)}(x,t) f(t) dt + \frac{1}{c_n(x)} f(x).$$

From this we conclude $L\tilde{u} = f$. The function

$$u(x) = \int_I g(x,t) f(t) dt$$

satisfies the equation $Lu = f$ since

$$u(x) = \tilde{u}(x) + \sum_{k=1}^{n} u_k(x) \int_I b_k(t) f(t) dt.$$

Finally we observe that

$$Ru = \int_{a^+}^{b^-} \underbrace{R(g(\cdot, t))}_{=0} f(t) dt$$

and the proof is completed. $\square$

The function g in Theorem 1.4 is called the **Green's function for the boundary value problem**

$$\begin{cases} Lu = f \\ Ru = 0. \end{cases}$$

Problem 1: Determine the Green's function for the boundary value problem

$$\begin{cases} -((1+x)u'(x))' = f(x),\ 0 \leq x \leq 1 \\ u'(0) = 0,\ u(1) = 0. \end{cases}$$

Solution: The functions $u_1(x) = 1$ and $u_2(x) = \ln(1+x)$ form a basis for the solutions to the homogeneous equation $-((1+x)u'(x))' = 0$. Note that

$$\begin{vmatrix} u_1'(0) & u_2'(0) \\ u_1(1) & u_2(1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & \ln 2 \end{vmatrix} = -1 \neq 0.$$

so there exists a Green's function. The fundamental solution $e(x,t) = a_1(t)u_1(x) + a_2(t)u_2(x)$ is given by

$$e(x,t) = a_1(t) + a_2(t)\ln(1+x)$$

and the constraints $e(t,t) = 0$, $e_x'(t,t) = -\frac{1}{1+t}$ easily yield

$$e(x,t) = \ln(1+t) - \ln(1+x).$$

The Green's function takes the form

$$g(x,t) = \theta(x-t)(\ln(1+t) - \ln(1+x)) + b_1(t) + b_2(t)\ln(1+x)$$

where

$$\begin{cases} g_x'(0,t) = 0 \\ g(1,t) = 0, \end{cases}$$

for $0 < t < 1$. Hence we get

$$\begin{cases} b_2(t) = 0 \\ \ln(1+t) - \ln 2 + b_1(t) + b_2(t)\ln 2 = 0 \end{cases}$$

from which we obtain

$$b_1(t) = \ln\frac{2}{1+t},\ b_2(t) = 0.$$

This finally gives

$$g(x,t) = \theta(x-t)\ln\frac{1+t}{1+x} + \ln\frac{2}{1+t}.$$

Problem 2: Assume that $\lambda \in \mathbb{C}$ and $f \in C([0,1])$. Show that the equation

$$\begin{cases} u''(x) + u'(x) + \lambda|u(x)| = f(x),\ 0 \leq x \leq 1 \\ u(0) = u(1) = 0,\ u \in C^2([0,1]) \end{cases}$$

has a unique solution for $|\lambda| < e(e-1)$.

Solution: We first determine the Green's function for the equation

$$\begin{cases} u'' + u' = F(x), \; 0 \le x \le 1 \\ u(0) = u(1) = 0. \end{cases}$$

The functions $u_1(x) = 1$ and $u_2(x) = e^{-x}$ form a basis for the solutions to the homogeneous equation $u'' + u' = 0$. With our standard notation we get

$$e(x,t) = 1 - e^{t-x}$$

and

$$g(x,t) = \theta(x-t)(1 - e^{t-x}) + \frac{e^t - e}{e-1} + \frac{e - e^t}{e-1} e^{-x}.$$

Note that

$$t > x \Rightarrow g(x,t) = \frac{e^t - e}{e-1}(1 - e^{-x}) \le 0$$

and

$$t \le x \Rightarrow g(x,t) = \frac{e^t - 1}{e-1}(1 - e^{1-x}) \le 0$$

which implies $g \le 0$.

For every $u \in C([0,1])$ define

$$(Tu)(x) = \int_0^1 g(x,t)(f(t) - \lambda |u(t)|)dt, \; 0 \le x \le 1$$

and observe that T maps $C([0,1])$ into $\{u \in C^2([0,1]); \, u(0) = u(1) = 0\}$. The equation in problem 2 has therefore a unique solution iff T has a unique fixed point. For $u, v \in C([0,1])$ it holds that

$$|(Tu)(x) - (Tv)(x)| = |\int_0^1 g(x,t)(\lambda|v(t)| - \lambda|u(t)|)dt| \le$$

$$\le |\lambda| \int_0^1 (-g(x,t)) ||v(t)| - |u(t)|| dt \le |\lambda| j(x) \|u - v\|_\infty,$$

where $\| \, \|_\infty$ denotes the max-norm for $C([0,1])$ and

$$j(x) = -\int_0^1 g(x,t)dt.$$

Since $j(0) = j(1) = 0$ and $j'' + j' = -1$ it follows that

$$j(x) = \frac{e}{e-1} - x - \frac{e}{e-1} e^{-x}$$

and

$$\max_{[0,1]} j = j\left(\ln \frac{e}{e-1}\right) = \frac{1}{e-1} + \ln\left(1 - \frac{1}{e}\right) \le$$

$$\le \frac{1}{e-1} - \frac{1}{e} = \frac{1}{e(e-1)}.$$

We conclude that

$$\|Tu - Tv\|_\infty \le \frac{|\lambda|}{e(e-1)} \|u - v\|_\infty$$

and Banach's fixed point theorem ([2]) implies that T has a unique fixed point for $|\lambda| < e(e-1)$.

1.3 Spectral theory for ordinary differential equations

The linear mapping $L_0 : C^n_R(I) \to C(I)$ is called **symmetric** if

$$\langle L_0 u, v\rangle = \langle u, L_0 v\rangle, \quad \text{all} \;\; u, v \in C^n_R(I),$$

where the inner product is given by the inner product in $L^2(I)$

$$\langle f, h\rangle = \int_a^b f(x)\overline{h(x)}dx.$$

Provided that L_0 is a bijection and g is the Green's function for the boundary value problem

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases},$$

we define

$$(Gf)(x) = \int_a^b g(x,t)f(t)dt, \; f \in C(I)$$

and

$$(\tilde{G}f)(x) = \int_a^b g(x,t)f(t)\,dt, \; f \in L^2(I).$$

Theorem 1.5. *Assume that L_0 is a bijection. Then the following conditions are equivalent:*

1. *L_0 is symmetric*
2. *$\tilde{G}$ is self-adjoint*
3. *$g(x,t) = \overline{g(t,x)}$, $x, t \in I$.*

Proof. (1) $\Leftrightarrow$ (2): L_0 is symmetric iff

$$\langle L_0 Gf, Gh\rangle = \langle Gf, L_0 Gh\rangle, \; f, h \in C(I)$$

which is the same as

$$\langle f, Gh\rangle = \langle Gf, h\rangle, \; f, h \in C(I).$$

This is equivalent to

$$\langle f, \tilde{G}h\rangle = \langle \tilde{G}f, h\rangle, \; f, h \in L^2(I)$$

since $C(I)$ is dense in $L^2(I)$ and $\tilde{G}$ is a bounded linear operator on $L^2(I)$ ([1] example 4.2.4) whose restriction to $C(I)$ is equal to G. L_0 being symmetric is thus equivalent to $\tilde{G}$ being self-adjoint.

(2) $\Leftrightarrow$ (3): We first observe that

$$(\tilde{G}^* f)(x) = \int_a^b \overline{g(t,x)}f(t)dt$$

([1] example 4.4.3). This implies that $\tilde{G} = \tilde{G}^*$ iff

$$\int_a^b (g(x,t) - \overline{g(t,x)}) f(t) dt = 0,\ f \in L^2(I).$$

Since g is continuous this means that $g(x,t) - \overline{g(t,x)} = 0$ for all $x, t \in I$ and so $g(x,t) = \overline{g(t,x)}$ for all $x, t \in I$. $\square$

Example 1: Consider the boundary value problem

$$\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0,\ 0 \le x \le 1. \end{cases}$$

This means that $Lu = -u''$, $R_1 u = u(0)$ and $R_2 u = u(1)$. The operator L_0 is symmetric since

$$\langle L_0 u, v\rangle = \int_0^1 -u''\bar{v}dx = \big[-u'\bar{v}\big]_0^1 + \int_0^1 u'\bar{v}'dx = \{Ru = 0\} =$$

$$= \langle u', v'\rangle = \overline{\langle v', u'\rangle} = \overline{\langle L_0 v, u\rangle} = \langle u, L_0 v\rangle$$

for all $u, v \in C_R^2([0,1])$. This fact also follows from Theorem 1.5 by checking that L_0 is a bijection and that the Green's function is given by

$$g(x,t) = \begin{cases} t(1-x), & 0 \le t < x \le 1 \\ (1-t)x, & 0 \le x \le t \le 1. \end{cases}$$

It easily follows that $g(x,t) = \overline{g(t,x)}$. The details are left as an exercise.

Theorem 1.6. *Assume that L_0 is symmetric and is a bijection. Then the following statements are true:*

1. *0 is not an eigenvalue for L_0 nor for $\tilde{G}$.*
2. *f is an eigenfunction for L_0 corresponding to the eigenvalue μ iff f is an eigenvalue for $\tilde{G}$ corresponding to the eigenvalue $1/\mu$.*

Proof. (1): $\mathcal{N}(L_0) = \{\mathbf{0}\}$ implies that L_0 has no eigenfunction corresponding to an eigenvalue zero.

Now assume that $f \in \mathcal{N}(\tilde{G})$. We will show that $f = \mathbf{0}$. For this take an arbitrary $\phi \in C_R^n(I)$. We obtain

$$\begin{aligned} 0 =& \langle \mathbf{0}, L_0\phi\rangle = \langle \tilde{G}f, L_0\phi\rangle = \langle f, \tilde{G}L_0\phi\rangle = \\ =& \langle f, GL_0\phi\rangle = \langle f, \phi\rangle. \end{aligned}$$

Since $C_R^n(I)$ is dense in $L^2(I)$ we can conclude that $f = \mathbf{0}$.

(2): $\Rightarrow$) From

$$\mathbf{0} \neq f = G(L_0 f) = G(\mu f) = \mu G f = \mu \tilde{G} f$$

it follows that f is an eigenfunction to $\tilde{G}$ corresponding to the eigenvalue $1/\mu$.

$\Leftarrow$) We have

$$\int_a^b g(x,t) f(t) dt = \frac{1}{\mu} f(x) \quad \text{a.e. in } I.$$

Setting

$$h(x) = \mu \int_a^b g(x,t) f(t) dt,\ x \in I$$

it follows from Lebesgue's dominated convergence theorem (see [3]) that $h \in C(I)$. Moreover we have $h(x) = f(x)$ *a.e.* in I and

$$h(x) = \mu \int_a^b g(x,t) h(t) dt,\ x \in I,$$

and hence we get $Gh = \frac{1}{\mu} h$. This yields

$$h = L_0(Gh) = L_0\left(\frac{1}{\mu} h\right) = \frac{1}{\mu} L_0 h.$$

Since $h \neq \mathbf{0}$ in $C_R^n(I)$, h is an eigenfunction to L_0 corresponding to the eigenvalue μ. Thus h, which is equal to f in $L^2(I)$, is an eigenfunction to L_0 corresponding to the eigenvalue μ. This is the proper interpretation of the formulation in Theorem 1.6 2) and the proof of the theorem is complete. $\square$

Theorem 1.7. *Assume that L_0 is symmetric and is a bijection. Moreover let $(\mu_n)_1^\infty$ denote the eigenvalues for L_0 counted with multiplicity and assume that $(e_n)_1^\infty$ is a corresponding sequence of orthonormal eigenfunctions. Then $(e_n)_1^\infty$ is an ON-basis for $L^2(I)$ and the solution to the equation*

$$\begin{cases} Lu = f \\ Ru = 0 \end{cases},$$

where $f \in C(I)$, is given by

$$u = \sum_1^\infty \frac{1}{\mu_n} \langle f, e_n \rangle e_n \quad (\text{in } L^2(I)).$$

Proof. The operator $\tilde{G}$ is compact ([1] example 4.8.4) and the Hilbert-Schmidt theorem ([1] theorem 4.10.1) and Theorem 1.6 1) implies that $(e_n)_1^\infty$ is a complete ON-sequence for $L^2(I)$. From

$$f = \sum_1^\infty \langle f, e_n \rangle e_n$$

in $L^2(I)$, Theorem 1.6 2) now implies that

$$u = Gf = \tilde{G} f = \sum_1^\infty \langle f, e_n \rangle \tilde{G} e_n = \sum_1^\infty \frac{1}{\mu_n} \langle f, e_n \rangle e_n$$

in $L^2(I)$. $\square$

Example 2: Consider the boundary value problem

$$\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0, \; 0 \le x \le 1. \end{cases}$$

Example 1 shows that the corresponding operator L_0 is symmetric and is a bijection. The eigenfunctions for L_0 are obtained as the non-trivial solutions to the equation

$$\begin{cases} -e''(x) = \mu e(x) \\ e(0) = e(1) = 0, \; 0 \le x \le 1 \end{cases}$$

and a simple calculation gives $e_n(x) = A \sin n\pi x$, where $A \neq 0$ and $n = 1, 2, \ldots$. The sequence $(\sqrt{2} \sin n\pi x)_1^\infty$ is therefore an ON-basis for $L^2([0,1])$.

Example 3: Wirtinger's inequality states that

$$\int_0^1 |u'(x)|^2 dx \ge \pi^2 \int_0^1 |u(x)|^2 dx$$

for all $u \in C^1([0,1])$ that satisfies $u(0) = u(1) = 0$. To show this we first let

$$u(x) = \sum_1^\infty a_n \sqrt{2} \sin n\pi x \quad (\text{in } L^2([0,1]))$$

where

$$a_n = \int_0^1 u(x) \sqrt{2} \sin n\pi x dx.$$

Furthermore we have

$$\int_0^1 u'(x)\sqrt{2}\cos n\pi x dx = \Big[u(x)\sqrt{2}\cos n\pi x\Big]_0^1 + \\ + n\pi \int_0^1 u(x)\sqrt{2} \sin n\pi x dx = n\pi a_n$$

and using the fact that the sequence $(\sqrt{2}\cos n\pi x)_1^\infty$ is an ON sequence, Bessel's inequality ([1] theorem 3.7.2) yields the estimate

$$\int_0^1 |u'(x)|^2 dx \ge \sum_1^\infty n^2 \pi^2 |a_n|^2$$

where the RHS is greater than or equal to

$$\pi^2 \sum_1^\infty |a_n|^2 = \pi^2 \int_0^1 |u(x)|^2 dx.$$

This gives one proof for Wirtinger's inequality.

References

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