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1. Consider the BVP

$$\begin{cases} Lu \equiv u''(x) + u(x) = -\lambda \cos(1 + u(x)), & x \in [0,1] \\ u(0) = u'(0) = 0,\ u \in C^2([0,1]) \end{cases} \qquad (*)$$

Calculation of the Green's function:

$$g(x,t) = (a_1(t)\cos x + a_2(t)\sin x)\theta(x-t) + b_1(t)\cos x + b_2(t)\sin x$$

where

$$\begin{cases} a_1(t)\cos t + a_2(t)\sin t = 0 \\ -a_1(t)\sin t + a_2(t)\cos t = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = -\sin t \\ a_2(t) = \cos t \end{cases}$$

and

$$\begin{cases} b_1(t) = 0 \\ b_2(t) = 0 \end{cases}$$

Hence we have $g(x,t) = \sin(x-t)\theta(x-t)$.

Now set

$$T : C([0,1]) \longrightarrow C([0,1]),$$

where $Tu(x) = \int_0^1 g(x,t)(-\lambda\cos(1+u(t)))\,dt$. From Banach's fixed point theorem we conclude that $(*)$ has a unique solution if T is a contraction. For $u, v \in C([0,1])$ we have

$$|Tu(x) - Tv(x)| = |\lambda|\,|\int_0^1 g(x,t)(\cos(1+u(t)) - \cos(1+v(t)))\,dt| \leq$$
$$\leq |\lambda| \int_0^1 |g(x,t)|dt\,\|u-v\|_\infty = |\lambda| \int_0^x \sin(x-t)\,dt\,\|u-v\|_\infty \leq$$
$$\leq |\lambda|\,(1-\cos 1)\,\|u-v\|_\infty.$$

Hence T is a contraction for $|\lambda| < \frac{1}{1-\cos 1}$ and the desired conclusion follows.

2. $(e_n)_{n=1}^\infty$ is an ON-basis in a Hilbert space H and T is defined by

$$T(\Sigma_{n=1}^\infty a_n e_n) = \Sigma_{n=1}^\infty \frac{1}{n+1} a_{n+1} e_n$$

for $(a_n)_{n=1}^\infty \in l^2$. Clearly $T \in \mathcal{B}(H,H)$ with $\|T\| = \frac{1}{2}$. An easy calculation gives

$$T^*(\Sigma_{n=1}^\infty b_n e_n) = \Sigma_{n=2} \frac{1}{n} b_{n-1} e_n.$$

T is a compact operator since $\|T - T_M\| \to 0$ as $M \to \infty$ where T_M, $M = 1, 2, \ldots$, are finite dimensional operators defined by

$$T_M(\Sigma_{n=1}^{\infty} a_n e_n) = \Sigma_{n=1}^{M} \frac{1}{n+1} a_{n+1} e_n.$$

More precisely we have $\|T - T_M\| < \frac{1}{M},\ M = 1, 2, \ldots$.
Moreover λ is an eigenvalue for T iff there exists an (eigen)vector $\mathbf{0} \neq \Sigma_{n=1}^{\infty} a_n e_n$ such that $T(\Sigma_{n=1}^{\infty} a_n e_n) = \lambda \Sigma_{n=1}^{\infty} a_n e_n$, i.e.

$$\lambda a_n = \frac{1}{n+1} a_{n+1}, \quad n = 1, 2, \ldots$$

This implies that only $\lambda = 0$ is an eigenvalue for T (with eigenvector e_1). Finally μ is an eigenvalue for T^* iff there exists an (eigen)vector $\mathbf{0} \neq \Sigma_{n=1}^{\infty} b_n e_n$ such that $T^*(\Sigma_{n=1}^{\infty} b_n e_n) = \mu \Sigma_{n=1}^{\infty} b_n e_n$. This means that

$$b_1 = 0,\ \mu b_n = \frac{1}{n} b_{n-1},\ n = 2, 3, \ldots$$

Hence T^* has no eigenvalues. This gives $\sigma_p(T) = \{0\}$ and $\sigma_p(T^*) = \emptyset$.

3. Riesz representation theorem implies that there are uniquely defined $y_k \in H$, $k = 1, 2, \ldots, n$, such that

$$f_k(x) = \langle x, y_k \rangle \ \text{ all } x \in H,$$

where H is a Hilbert space. Moreover the fact that $f_1, f_2, \ldots, f_n$ are linearly independent in $\mathcal{B}(H, \mathbb{C})$ implies that $y_1, y_2, \ldots, y_n$ are linearly independent[1] in H (easy to show). Now for each $l \in \{1, 2, \ldots, n\}$ consider the set $Y_l = \{y_k : k \neq l\}^{\perp}$. We see that $f_l|_{Y_l} \neq \mathbf{0}$ since otherwise $f_l(x) = 0$ for all $x \in Y_l$. This would imply that

$$\{y_k : k \neq l\}^{\perp} \subset \{y_l\}^{\perp}$$

and hence

$$\text{Span}\{y_l\} \subset \text{Span}\{y_k : k \neq l\},$$

which contradicts the linearly independence of $y_1, y_2, \ldots, y_n$ Finally, for each $l \in \{1, 2, \ldots, n\}$ pick an $x_l \in Y_l$ such that $f_l(x_l) = 1$. These x_l:s will satisfy the properties stated in the problem.

4. See textbook

5. See textbook

[1] $y_1, y_2, \ldots, y_n$ does not need to be pairwise orthogonal.

6. Let H be a Hilbert space and let $T : H \to H$ be a linear mapping with the following property:

$$x_n \to x \text{ in } H \quad \Rightarrow \quad Tx_n \rightharpoonup Tx \text{ in } H.$$

We should prove that T is bounded[2].

Assume that T is not bounded. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to \mathbf{0}$ in H but $Tx_n \not\rightharpoonup T\mathbf{0} = \mathbf{0}$ in H. Without loss of generality (easy to show) we may assume that

(a) $\|Tx_n\| = 1$ for $n = 1, 2, \ldots$,

(b) $\|x_n\| \leq 2^{-n}$ for $n = 1, 2, \ldots$,

(c) $Tx_n \rightharpoonup \mathbf{0}$ in H.

Set $y_n = Tx_n$ for $n = 1, 2, \ldots$ and choose an increasing sequence $(n_l)_{l=1}^{\infty}$ of integers as follows: Set $n_1 = 1$. For $l = 2, 3, \ldots$ let n_l have the property

$$\Sigma_{k=1}^{l-1} |\langle y_{n_k}, y_m \rangle| \leq \frac{1}{4} \text{ all } m \geq n_l.$$

The existence of $(n_l)_{l=1}^{\infty}$ follows from the fact that $y_n \rightharpoonup \mathbf{0}$. This implies that

$$\|\Sigma_{l=1}^{M} y_{n_l}\|^2 = \Sigma_{l=1}^{M} \langle y_{n_l}, y_{n_l} \rangle + \Sigma_{k,l=1,k\neq l}^{M} \langle y_{n_k}, y_{n_l} \rangle \geq$$

$$\geq M - 2\Sigma_{1\leq k<l\leq M} |\langle y_{n_k}, y_{n_l} \rangle| = \Sigma_{l=1}^{M} (1 - 2\Sigma_{k=1}^{l-1} |\langle y_{n_k}, y_{n_l} \rangle|) \geq \frac{M}{2}.$$

Now $y_{n_k} = Tx_{n_k}$ and $\|x_{n_k}\| \leq 2^{-k}$. Hence $\Sigma_{k=1}^{M} x_{n_k} \to \tilde{x}$ for some $\tilde{x} \in H$ but $T(\Sigma_{k=1}^{M} x_{n_k}) = \Sigma_{k=1}^{M} Tx_{n_k} \not\rightharpoonup T\tilde{x}$ since[3] $\|\Sigma_{k=1}^{M} Tx_{n_k}\|^2 \geq \frac{M}{2} \to \infty$ as $M \to \infty$. Contradiction! Hence T is bounded.

[2] This can be done using the closed graph theorem, see the lecture notes on spectral theory, but I have not discussed that theorem in class.

[3] Every weakly convergent sequence is bounded