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1. Consider the BVP

$$\begin{cases} Lu = u''(x) - u(x) = -\frac{1}{2}(1 + u(x^2)), & x \in [0,1] \\ u(0) = u'(0) = 0,\ u \in C^2([0,1]) \end{cases} \qquad (*)$$

Calculation of Green's function

$$g(x,t) = (a_1(t)e^x + a_2(t)e^{-x})\theta(x-t) + b_1(t)e^x + b_2(t)e^{-x}$$

where

$$\begin{cases} a_1(t)e^t + a_2(t)e^{-t} = 0 \\ a_1(t)e^t - a_2(t)e^{-t} = 1 \end{cases} \quad \text{i.e.} \quad \begin{array}{l} a_1(t) = \frac{1}{2}e^{-t} \\ a_2(t) = -\frac{1}{2}e^{t} \end{array}$$

and

$$\begin{cases} b_1(t) + b_2(t) = 0 \\ b_1(t) - b_2(t) = 0 \end{cases} \quad \text{i.e.} \quad \begin{array}{l} b_1(t) = 0 \\ b_2(t) = 0 \end{array}$$

Hence we have $g(x,t) = \sinh(x-t)\theta(x-t)$.

Now set

$$T : C([0,1]) \longrightarrow C([0,1]),$$

where $Tu(x) = \int_0^1 g(x,t)(-\frac{1}{2}(1 + u(t^2)))dt$. From Banach's fixed point theorem we conclude that $(*)$ has a unique solution if T is a contraction. For $u, v \in C([0,1])$ we have

$$|Tu(x) - Tv(x)| = |\int_0^1 g(x,t)\frac{1}{2}(u(t^2) - v(t^2))dt| \leq$$

$$\leq \frac{1}{2}\int_0^1 |g(x,t)|dt\|u-v\|_\infty = \frac{1}{2}(\cosh k - 1)\|u-v\|_\infty \leq$$

$$\leq \underbrace{\frac{1}{4}(e + \frac{1}{e} - 2)}_{<1}\|u-v\|_\infty.$$

Here T is a contraction and the statement follows.

2. Set, for $f \in L^2([a,b]), Tf(x) = \frac{1}{b-a}\int_a^b f(x)dx,\ x \in [a,b]$.

$Tf \in L^2([a,b])$ since

$$\|f\|_{L^2}^2 = \int_a^b (\frac{1}{b-a}|\int_a^b f(x)\,dx|)^2 dt =$$

$$= (\frac{1}{b-a})^2 \int_a^b |\int_a^b f(x)dx|^2 dt \leq \{\text{Hölder}\} \leq$$

$$\leq (\frac{1}{b-a})^2 \int_a^b (b-a)\int_a^b |f(x)|^2 dx dt = \int_a^b |f(x)|^2 dt = \|f\|_{L^2}^2.$$

T linear: easy to show.

T bounded: see above. In particular we get $\|T\| \leq 1$.

To show that T is an orthogonal projection it suffices to show that $T^2 = T$ and $T^* = T$.

(a) Take $f \in L^2([a,b])$. Then

$$(T^2 f)(x) = T(\frac{1}{b-a}\int_a^b f(t)dt) = \frac{1}{b-a}\int_a^b \frac{1}{b-a}\int_a^b f(x)dtds =$$
$$= \frac{1}{b-a}\int_a^b f(t)dt = (Tf)(x), \quad \text{all} \quad x \in [a,b].$$

Hence $T^2 = T$.

(b) Take $f, g \in L^2([a,b])$. We obtain

$$\begin{aligned}\langle Tf, g\rangle &= \int_a^b \frac{1}{b-a}\int_a^b f(x)dx \cdot \overline{g(t)}dt = \\ &= \int_a^b f(x)\frac{1}{b-a}\int_a^b \overline{g(t)}dtdx = \\ &= \int_a^b f(x)\,\overline{\frac{1}{b-a}\int_a^b g(t)dt}dx = \langle f, Tg\rangle\end{aligned}$$

Hence $T^* = T$.

The statement is proved.

3. Let $h \in C([0,1] \times [0,1])$ be real-valued and

$$h(x,y) = h(y,x) > 0 \quad \text{all} \quad x, y \in [0,1]. \tag{$*$}$$

Set $Tf(x) = \int_0^1 h(x,y)f(y)dy$, $x \in [0,1]$, for $f \in L^2([0,1])$. We want to show that T has an eigenvalue $\lambda = \|T\|$ whitch is simple. (All eigenvalues λ satisfy $|\lambda| \leq \|T\|$). Since the kernel is continuous and satisfies $(*)$ we see that T is a compact, self-adjoint operator or $L^2([0,1])$ and hence has an eigenvalue $\lambda \in \mathbb{R}$ with $|\lambda| = \|T\|$. Since $h > 0$ we see that $\lambda = \|T\|$ (see first and second observation below). It remains to prove that this eigenvalue is simple.

First observation: f eigenfunction for $\lambda \Rightarrow f \in C([0,1])$ which follows from Lebesgues dominated convergence. Then
Second observation: f eigenfunction for $\lambda \Rightarrow f$ has constant sign, say $f \geq 0$, since if f changes sign, then

$$\lambda\|f\| = \|T\|\|f\| = \|Tf\| < \|T|f|\| \leq \|T\|\,\||f|\| = \|T\|\,\|f\|.$$

Moreover we can conclude that $f > 0$ since $h > 0$.

Third observation: f_1, f_2 eigenfunction for $\lambda \Rightarrow f_1 = \alpha f_2$ for some $\alpha \neq 0$.

To see this assume that it is false and set

$$s(\alpha) = |\{x \in [0,1] : f_1(x) - \alpha f_2(x) \geq 0\}|,\ \alpha \geq 0,$$

where $|E|$ denotes the measure of the set E. Here $s(0) = 1$, $s(\alpha)$ decreasing and $\lim_{a \to +\infty} s(\alpha) = 0$. Also $s(\alpha) = 1$ for $\alpha \in [0, \tilde{\alpha}]$ for some $\tilde{\alpha} > 0$ since $f_1, f_2 \in C([0,1])$ and $f_1, f_2 > 0$ on $[0,1]$. Then there exists $0 < \alpha_0 < \alpha_1$ such that $1 > s(\alpha_0) \geq s(\alpha_1) > 0$. *)

This means that

$$\begin{aligned} f_1(x) &< \alpha_0 f_1(x) && \text{on a set of positive measure} \\ f_1(x) &\geq \alpha_0 f_2(x) && \text{on a set of positive measure} \end{aligned}$$

but since also

$$f_1(x) \geq \alpha_1 f_2(x) \qquad \text{on a set of positive measure,}$$

together with $0 < \alpha_0 < \alpha_1$ and $f_2 > 0$ we see that

$$f_1(x) > \alpha_0 f_2(x) \qquad \text{on a set of positive measure.}$$

Then $f = f_1 - \alpha_0 f_2$ is an eigenfunction for λ (note that $f \neq 0$) that changes sign. Contradiciton!

The statement follows.

*) It cannot happen that $\exists \beta > 0$ sth.

$$\begin{cases} s(\alpha) = 1 & \text{for} \quad \alpha < \beta \\ s(\alpha) = 0 & \text{for} \quad \alpha > \beta \end{cases} \qquad \text{easy to see.}$$

and if there exists a $\beta > 0$ sth

$$\begin{aligned} s(\alpha) = 1 \quad &\text{for} \quad \alpha \leq \beta \\ s(\alpha) = 0 \quad &\text{for} \quad \alpha > \beta \end{aligned}$$

then $f_1 = \beta f_2$.