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1. Consider the BVP

$$\begin{cases} u''(x) = \frac{1}{5}(1 - \frac{1}{1+u^4(x)}), & x \in [0,1] \\ u(0) = u(1) = 0,\ u \in^2 ([a,1]) \end{cases} \qquad (*)$$

Step 1: Calculate the Green's function $g(x,t)$ for $Lu = u''$, $u(0) = u(1) = 0$. Here

$$g(x,t) = (a_1(t) \cdot 1 + a_2(t)x)\theta(x-t) + b_1(t) \cdot 1 + b_2(t)x$$

where

$$\begin{cases} a_1(t) + a_2(t)t = 0 \\ a_2(t) = 1 \end{cases} \quad \text{i.e.} \quad \begin{matrix} a_1(t) = -t \\ a_2(t) = 1 \end{matrix}$$

and

$$\begin{cases} g(0,t) = 0,\ 0 < t < 1 \\ g(1,t) = 0,\ 0 < t < 1 \end{cases} \quad \text{i.e.} \quad \begin{matrix} b_1(t) = 0 \\ b_2(t) = t - 1 \end{matrix}$$

which yields

$$g(x,t) = \begin{cases} (t-1)x & 0 \le x < t \le 1 \\ (x-1)t & 0 \le t < x \le 1 \end{cases}$$

Step 2: Set

$$Tu(x) = \int_0^1 g(x,t)(\frac{1}{5} - \frac{1}{5}\ \frac{1}{1+u^4(t)})dt,\ u \in C([0,1]).$$

If $T : C([0,1]) \to C([0,1])$ is a contraction, then $(*)$ has a unique solution. T is a contraction since for $u, v \in C([0,1])$,

$$|Tu(x) - Tv(x)| \le \frac{1}{5}\int_0^1 |g(x,t)|\,|\frac{1}{1+u^4(t)} - \frac{1}{1+v^4(t)}|dt \le$$
$$\le \{\text{mean value thm }\}| \le \ldots \le \frac{4}{5}\int_0^1 |g(x,t)|dt \|u - v\|_\infty \le$$
$$\le \frac{4}{10}\|u - v\|_\infty.$$

Hence $\|Tu - Tv\|_\infty \le \frac{4}{10}\|u - v\|_\infty$ and so T is a contraction. The Banach fixed point theorem yields the result.

2. $(e_n)_{n=1}^\infty$ ON-basis on H implies $(f_k)_{k=-\infty}^\infty$ is an ON-basis on H where

$$f_0 = e_1,\ f_k = e_{2k+1},\ k = 1, 2, \ldots,\ f_k = e_{2k},\ k = 1, 2, \ldots,$$

so every $x \in H$ has a unique representation $\sum_{k=-\infty}^{\infty} a_k f_k$, where $\sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$ (more precisely $a_k = \langle x, f_k \rangle, \quad k \in \mathbb{Z}$).

Clearly, S is well-defined and

$$\|S(\sum_{k=-\infty}^{\infty} a_k f_k)\|^2 = \|\sum_{k=-\infty}^{\infty} a_k f_{k+1}\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\|\sum_{k=-\infty}^{\infty} a_k f_k\|^2 = \sum_{k-\infty}^{\infty} |a_k|^2$$

by Parseval's formula. Hence S is an isometry and $\|S\| = 1$. S has no eigenvalues since if

$\lambda = 0$ then $a_k = 0$ all k

$\lambda \neq 0$ then $a_k = \lambda a_{k+1}$ all k and hence $\displaystyle\sum_{k=-\infty}^{\infty} |a_k|^2 = \infty$

unless $\sum_{k=-\infty}^{\infty} |a_k|^2 = 0$.

3. $(e_k)_{k=1}^{\infty}$ is a sequence in H, where $\|e_k\| = 1$ for all k. We want to show that

$$\sum_k |\langle x, e_k \rangle|^2 \leq \|x\|^2 (1 + (\sum_{k \neq \ell} |\langle e_k, e_\ell \rangle|^2)^{1/2}).$$

Note that

$$\sum_k |\langle x, e_k \rangle|^2 = \sum_k \langle x, e_k \rangle \overline{\langle x, e_k \rangle} = \langle x, \Sigma_k \langle x, e_k \rangle e_k \rangle \leq$$
$$\leq \|x\| \cdot \|\Sigma_k \langle x, e_k \rangle e_k\|.$$

Moreover

$$\|\sum_k \langle x, e_k \rangle e_k\|^2 = \sum_{k,\ell} \langle x, e_k \rangle \overline{\langle x, e_\ell \rangle} \langle e_k, e_\ell \rangle \leq$$
$$\leq \sum_{k,\ell} |\langle x, e_k \rangle| |\langle x, e_\ell \rangle| |\langle e_k, e_\ell \rangle| =$$
$$= \sum_k |\langle x, e_k \rangle|^2 + \sum_{k \neq \ell} (|\langle x, e_k \rangle| |\langle x, e_\ell \rangle|) |\langle e_k, e_\ell \rangle| \leq$$

$$\leq \sum_k |\langle x, e_k\rangle|^2 + (\sum_{k\neq\ell}(|\langle x, e_k\rangle|^2 \cdot |\langle x, e_\ell\rangle|^2)^{1/2} \times$$

$$\times (\sum_{k\neq\ell} |\langle e_k, e_\ell\rangle|^2)^{1/2} \leq$$

$$\leq \sum_k |\langle x, e_k\rangle|^2 + (\sum_{k,\ell} |\langle x, e_k\rangle|^2 |\langle x, e_\ell\rangle|^2)^{1/2} \times$$

$$\times (\sum_{k\neq\ell} |\langle e_k, e_\ell\rangle|^2)^{1/2} =$$

$$= \sum_k |\langle e_k, e_\ell\rangle|^2 (1 + (\sum_{k\neq\ell} |\langle e_k, e_\ell\rangle|^2)^{1/2}).$$

This yields the result.

4. 5.86. Theory from book (and lectures).