Matematik, CTH & GU
Tentamen i Funktionalanalys TMA401/MAN670
Hjälpmedel: Inga (inte ens räknedosa).
Personuppgifter: Namn, personnummer, linje, antagningsår.
Inlämning ska ske i uppgifternas ordning; v.g. sidnumrera!
Teoriuppgifter: 4,5,6.
Telefon: Rolf Liljendahl 073-9979268

Datum: 2004–08–28
Skrivtid: fm (5 timmar)

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1. Show that the following boundary value problem

$$\begin{cases} 5u''(x) + \dfrac{1}{1+u(x)^4} = 1, \quad 0 \le x \le 1 \\ u(0) = u(1) = 0, \quad u \in C^2([0,1]) \end{cases}$$

has a unique solution.

(4p)

2. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space H and set

$$\begin{cases} f_0 = e_1 \\ f_k = e_{2k+1} \quad k > 0 \\ f_k = e_{-2k} \quad k < 0 \end{cases}.$$

Moreover define S by $S(\Sigma_{k=-\infty}^{\infty} a_k f_k) = \Sigma_{k=-\infty}^{\infty} a_k f_{k+1}$. Show that S is a well-defined bounded linear mapping on H, calculate $\|S\|$ and show that S has no eigenvalues.

(4p)

3. Let $(e_k)_{k=1}^{n}$ be a sequence of vectors in a Hilbert space H. Assume that $\|e_k\| = 1$ for all k. Show[1] that

$$\Sigma_{k=1}^{n} |\langle x, e_k\rangle|^2 \le \|x\|^2 (1 + (\Sigma_{\substack{k,l=1 \\ k \ne l}}^{n} |\langle e_k, e_l\rangle|^2)^{\frac{1}{2}})$$

for all $x \in H$. Note that if $(e_k)_{k=1}^{n}$ is an ON-sequence in H then the statement is called Bessel's inequality.

(4p)

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[1]Hint: Note that $\Sigma|\langle x, e_k\rangle|^2 = \langle x, \Sigma\langle x, e_k\rangle e_k\rangle$.

4. State and prove the Lax-Milgram theorem.

(5p)

5. Let P, Q be orthogonal projections on a Hilbert space such that $PQ = QP$. Show that $P + Q - PQ$ is an orthogonal projection.

(4p)

6. Let $T : H \to H$ be a compact linear operator on a Hilbert space H. Show that $\mathcal{R}(I + T)$ is a closed subspace of H.

(4p)

Good Luck!!
PK