CHALMERS | GÖTEBORGS UNIVERSITET

Matematiska Institutionen
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10th April 2006

TMA401 Functional Analysis
MAN670 Applied Functional Analysis
4th quarter 2005/2006

All document concerning the course can be found on the course home page:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/

Home Assignment 1

Problem 1: Let $(E, \|\cdot\|)$ be a normed space and let x, y be two vectors in E. Does

$$\|x+y\| = \|x\| + \|y\|$$

imply that one of the vectors is a scalar multiple of the other? Answer the question for l^1 and l^2.

Problem 2: Assume that $p \in (1, \infty)$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$\|\mathbb{x}\|_{l^p} = \sup\{\Sigma_{n=1}^{\infty} |x_n y_n| : \mathbb{y} \in l^q \text{ with } \|\mathbb{y}\|_{l^q} = 1\}.$$

Problem 3: Show that for every $g \in C([0,1])$ the mapping $T : C([0,1]) \to \mathbb{R}$ defined by

$$Tf = \int_0^1 f(t) g(t)\, dt$$

is linear and bounded. Moreover, calculate the operator norm $\|T\|$. (All functions are considered to be real-valued and $C([0,1])$ is equipped with the sup-norm)

Problem 4: Consider the vector space l^1 and set $\|\mathbb{x}\|_* = 2|\Sigma_{n=1}^{\infty} x_n| + \Sigma_{n=2}^{\infty}(1+\frac{1}{n})|x_n|$ for $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^1$. Show that $\|\mathbb{x}\|_*$ defines a norm on l^1 and that the vector space l^1 is a Banach space with this norm. Is this norm equivalent to the standard norm $\|\mathbb{x}\|_{l^1}$?

Problem 5: Let $\mathcal{B}$ be a basis in an infinite-dimensional Banach space $(E, \|\cdot\|)$. Show that $\mathcal{B}$ cannot be countable.

Problem 6: Let F be a compact set in a normed space $(E, \|\cdot\|)$ and let $T : F \to F$ have the property

$$\|T(x) - T(y)\| < \|x - y\|, \quad \text{all } x, y \in F \text{ where } x \neq y.$$

Show that T has a unique fixed point.

The solutions should be handed in at the latest on **Friday April 28**.