



11th May 2006

## TMA401 Functional Analysis

### MAN670 Applied Functional Analysis

### 4th quarter 2005/2006

All document concerning the course can be found on the course home page:  
<http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/>

## Home Assignment 2

**Problem 1:** Let  $(x_n)_{n=1}^{\infty}$  be an ON-basis in a Hilbert space  $H$  and let  $(y_n)_{n=1}^{\infty}$  be another ON-sequence such that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1.$$

Show that  $(y_n)_{n=1}^{\infty}$  is an ON-basis.

**Problem 2:** Let  $x_n = (0, 0, \dots, 0, 1, 2, 0, \dots)$  where the numbers 1 and 2 appear in the positions  $n$  and  $n + 1$  and let  $y_n = (1, 1, \dots, 1, 0, 0, \dots)$  with the number 1 in the first  $n$  positions. Consider these as vectors in  $l^2$ . Prove that for all  $n = 1, 2, \dots$

$$y_n \notin \overline{\text{Span}\{x_1, x_2, \dots\}}.$$

**Problem 3:** Prove the existence and uniqueness of a solution to the following boundary value problem:

$$\begin{cases} 4u''(x) = |x + u(x)|, & 0 \leq x \leq 1 \\ u(0) - 2u(1) = u'(0) - 2u'(1) = 0, & u \in C^2([0, 1]). \end{cases}$$

**Problem 4:** Set  $H = L^2([0, 1])$ . Let  $T$  be given by

$$Tf(x) = \int_{1-x}^1 f(t) dt.$$

Show that

1.  $T$  is a bounded linear operator on  $H$  and
2. calculate the kernel  $k(x, t)$  for  $T$  and show that  $T$  is self-adjoint.
3. Moreover calculate the kernel  $k_2(x, t)$  for  $T^2$  and
4. find<sup>1</sup> all eigenvalues and eigenfunctions for  $T$ .
5. Finally calculate  $\|T\|$ .

The solutions should be handed in at the latest on **Friday May 19**.

<sup>1</sup>Hint: Let  $f$  be an eigenfunction for  $T$  and calculate  $(T^2 f)''$ . Show that  $f$  is a solution to the equation  $\lambda^2 f'' + f = 0$ .