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Matematiska Institutionen
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TMA401 Functional Analysis
MAN670 Applied Functional Analysis
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Gamla tentor från

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lösningsförslag till några tentor

Lösningsförslag till TMA401/MAN670 2005-05-25

1. Consider the BVP
$$\begin{cases} Lu \equiv u''(x) + u(x) = -\lambda\cos(1+u(x)), & x \in [0,1] \\ u(0) = u'(0) = 0,\ u \in C^2([0,1]) \end{cases} \qquad (*)$$
Calculation of the Green's function:
$$g(x,t) = (a_1(t)\cos x + a_2(t)\sin x)\theta(x-t) + b_1(t)\cos x + b_2(t)\sin x$$
where
$$\begin{cases} a_1(t)\cos t + a_2(t)\sin t = 0 \\ -a_1(t)\sin t + a_2(t)\cos t = 1 \end{cases} \quad \text{i.e.} \quad \begin{cases} a_1(t) = -\sin t \\ a_2(t) = \cos t \end{cases}$$
and
$$\begin{cases} b_1(t) = 0 \\ b_2(t) = 0 \end{cases}$$
Hence we have $g(x,t) = \sin(x-t)\theta(x-t)$.
Now set
$$T : C([0,1]) \longrightarrow C([0,1]),$$
where $Tu(x) = \int_0^1 g(x,t)(-\lambda\cos(1+u(t)))\,dt$. From Banach's fixed point theorem we conclude that $(*)$ has a unique solution if T is a contraction. For $u, v \in C([0,1])$ we have
$$\begin{aligned} &|Tu(x) - Tv(x)| = |\lambda|\,|\int_0^1 g(x,t)(\cos(1+u(t)) - \cos(1+v(t)))\,dt| \leq \\ &\leq |\lambda| \int_0^1 |g(x,t)|dt\,\|u-v\|_\infty = |\lambda| \int_0^x \sin(x-t)\,dt\,\|u-v\|_\infty \leq \\ &\leq |\lambda|\,(1-\cos 1)\,\|u-v\|_\infty. \end{aligned}$$
Hence T is a contraction for $|\lambda| < \frac{1}{1-\cos 1}$ and the desired conclusion follows.

2. $(e_n)_{n=1}^\infty$ is an ON-basis in a Hilbert space H and T is defined by
$$T(\Sigma_{n=1}^\infty a_n e_n) = \Sigma_{n=1}^\infty \frac{1}{n+1} a_{n+1} e_n$$
for $(a_n)_{n=1}^\infty \in l^2$. Clearly $T \in \mathcal{B}(H,H)$ with $\|T\| = \frac{1}{2}$. An easy calculation gives
$$T^*(\Sigma_{n=1}^\infty b_n e_n) = \Sigma_{n=2} \frac{1}{n} b_{n-1} e_n.$$
T is a compact operator since $\|T - T_M\| \to 0$ as $M \to \infty$ where T_M, $M = 1, 2, \ldots$, are finite dimensional operators defined by
$$T_M(\Sigma_{n=1}^\infty a_n e_n) = \Sigma_{n=1}^M \frac{1}{n+1} a_{n+1} e_n.$$
More precisely we have $\|T - T_M\| < \frac{1}{M}$, $M = 1, 2, \ldots$.
Moreover λ is an eigenvalue for T iff there exists an (eigen)vector $\mathbf{0} \neq \Sigma_{n=1}^\infty a_n e_n$ such that $T(\Sigma_{n=1}^\infty a_n e_n) = \lambda \Sigma_{n=1}^\infty a_n e_n$, i.e.
$$\lambda a_n = \frac{1}{n+1} a_{n+1}, \quad n = 1, 2, \ldots$$
This implies that only $\lambda = 0$ is an eigenvalue for T (with eigenvector e_1).
Finally μ is an eigenvalue for T^* iff there exists an (eigen)vector $\mathbf{0} \neq \Sigma_{n=1}^\infty b_n e_n$ such that $T^*(\Sigma_{n=1}^\infty b_n e_n) = \mu \Sigma_{n=1}^\infty b_n e_n$. This means that
$$b_1 = 0,\ \mu b_n = \frac{1}{n} b_{n-1},\ n = 2, 3, \ldots$$
Hence T^* has no eigenvalues. This gives $\sigma_p(T) = \{0\}$ and $\sigma_p(T^*) = \emptyset$.

3. Riesz representation theorem implies that there are uniquely defined $y_k \in H$, $k = 1, 2, \ldots, n$, such that
$$f_k(x) = \langle x, y_k \rangle \text{ all } x \in H,$$
where H is a Hilbert space. Moreover the fact that $f_1, f_2, \ldots, f_n$ are linearly independent in $\mathcal{B}(H, \mathbb{C})$ implies that $y_1, y_2, \ldots, y_n$ are linearly independent[1] in H (easy to show). Now for each $l \in \{1, 2, \ldots, n\}$ consider the set $Y_l = \{y_k : k \neq l\}^\perp$. We see that $f_l|_{Y_l} \neq \mathbf{0}$ since otherwise $f_l(x) = 0$ for all $x \in Y_l$. This would imply that
$$\{y_k : k \neq l\}^\perp \subset \{y_l\}^\perp$$
and hence
$$\text{Span}\{y_l\} \subset \text{Span}\{y_k : k \neq l\},$$
which contradicts the linearly independence of $y_1, y_2, \ldots, y_n$ Finally, for each $l \in \{1, 2, \ldots, n\}$ pick an $x_l \in Y_l$ such that $f_l(x_l) = 1$. These x_l:s will satisfy the properties stated in the problem.

4. See textbook

5. See textbook

6. Let H be a Hilbert space and let $T : H \to H$ be a linear mapping with the following property:
$$x_n \to x \text{ in } H \quad \Rightarrow \quad Tx_n \rightharpoonup Tx \text{ in } H.$$
We should prove that T is bounded[2].

Assume that T is not bounded. Then there exists a sequence $(x_n)_{n=1}^\infty$ such that $x_n \to \mathbf{0}$ in H but $Tx_n \not\rightharpoonup T\mathbf{0} = \mathbf{0}$ in H. Without loss of generality (easy to show) we may assume that

(a) $\|Tx_n\| = 1$ for $n = 1, 2, \ldots$,
(b) $\|x_n\| \leq 2^{-n}$ for $n = 1, 2, \ldots$,
(c) $Tx_n \rightharpoonup \mathbf{0}$ in H.

Set $y_n = Tx_n$ for $n = 1, 2, \ldots$ and choose an increasing sequence $(n_l)_{l=1}^\infty$ of integers as follows: Set $n_1 = 1$. For $l = 2, 3, \ldots$ let n_l have the property
$$\Sigma_{k=1}^{l-1} |\langle y_{n_k}, y_m \rangle| \leq \frac{1}{4} \text{ all } m \geq n_l.$$
The existence of $(n_l)_{l=1}^\infty$ follows from the fact that $y_n \rightharpoonup \mathbf{0}$. This implies that
$$\|\Sigma_{l=1}^M y_{n_l}\|^2 = \Sigma_{l=1}^M \langle y_{n_l}, y_{n_l} \rangle + \Sigma_{k,l=1,k\neq l}^M \langle y_{n_k}, y_{n_l} \rangle \geq$$
$$\geq M - 2\Sigma_{1\leq k<l\leq M} |\langle y_{n_k}, y_{n_l} \rangle| = \Sigma_{l=1}^M (1 - 2\Sigma_{k=1}^{l-1} |\langle y_{n_k}, y_{n_l} \rangle|) \geq \frac{M}{2}.$$
Now $y_{n_k} = Tx_{n_k}$ and $\|x_{n_k}\| \leq 2^{-k}$. Hence $\Sigma_{k=1}^M x_{n_k} \to \tilde{x}$ for some $\tilde{x} \in H$ but $T(\Sigma_{k=1}^M x_{n_k}) = \Sigma_{k=1}^M Tx_{n_k} \not\rightharpoonup T\tilde{x}$ since[3] $\|\Sigma_{k=1}^M Tx_{n_k}\|^2 \geq \frac{M}{2} \to \infty$ as $M \to \infty$. Contradiction! Hence T is bounded.

[1] $y_1, y_2, \cdots, y_n$ does not need to be pairwise orthogonal.
[2] This can be done using the closed graph theorem, see the lecture notes on spectral theory, but I have not discussed that theorem in class.
[3] Every weakly convergent sequence is bounded

Lösningsförslag till TMA401/MAN670 2004-08-28

1. Consider the BVP
$$\begin{cases} u''(x) = \frac{1}{5}(1 - \frac{1}{1+u^4(x)}), & x \in [0,1] \\ u(0) = u(1) = 0, \ u \in^2 ([a,1]) \end{cases} \qquad (*)$$

Step 1: Calculate the Green's function $g(x,t)$ for $Lu = u''$, $u(0) = u(1) = 0$. Here
$$g(x,t) = (a_1(t) \cdot 1 + a_2(t)x)\theta(x-t) + b_1(t) \cdot 1 + b_2(t)x$$
where
$$\begin{cases} a_1(t) + a_2(t)t = 0 \\ a_2(t) = 1 \end{cases} \quad \text{i.e.} \quad \begin{array}{l} a_1(t) = -t \\ a_2(t) = 1 \end{array}$$
and
$$\begin{cases} g(0,t) = 0, \ 0 < t < 1 \\ g(1,t) = 0, \ 0 < t < 1 \end{cases} \quad \text{i.e.} \quad \begin{array}{l} b_1(t) = 0 \\ b_2(t) = t - 1 \end{array}$$
which yields
$$g(x,t) = \begin{cases} (t-1)x & 0 \le x < t \le 1 \\ (x-1)t & 0 \le t < x \le 1 \end{cases}$$
Step 2: Set
$$Tu(x) = \int_0^1 g(x,t)(\frac{1}{5} - \frac{1}{5}\,\frac{1}{1+u^4(t)})dt, \ u \in C([0,1]).$$
If $T : C([0,1]) \to C([0,1])$ is a contraction, then $(*)$ has a unique solution. T is a contraction since for $u, v \in C([0,1])$,
$$|Tu(x) - Tv(x)| \le \frac{1}{5}\int_0^1 |g(x,t)|\,|\frac{1}{1+u^4(t)} - \frac{1}{1+v^4(t)}|dt \le$$
$$\le \{\text{mean value thm }\}| \le \ldots \le \frac{4}{5}\int_0^1 |g(x,t)|dt\|u-v\|_\infty \le$$
$$\le \frac{4}{10}\|u-v\|_\infty.$$
Hence $\|Tu - Tv\|_\infty \le \frac{4}{10}\|u-v\|_\infty$ and so T is a contraction. The Banach fixed point theorem yields the result.

2. $(e_n)_{n=1}^\infty$ ON-basis on H implies $(f_k)_{k=-\infty}^\infty$ is an ON-basis on H where
$$f_0 = e_1, \ f_k = e_{2k+1}, \ k = 1,2,\ldots, \ f_k = e_{2k}, \ k = 1,2,\ldots,$$
so every $x \in H$ has a unique representation $\sum_{k=-\infty}^\infty a_k f_k$, where $\sum_{k=-\infty}^\infty |a_k|^2 < \infty$ (more precisely $a_k = \langle x, f_k\rangle, \quad k \in \mathbb{Z}$).
Clearly, S is well-defined and
$$\|S(\sum_{k=-\infty}^\infty a_k f_k)\|^2 = \|\sum_{k=-\infty}^\infty a_k f_{k+1}\|^2 = \sum_{k=-\infty}^\infty |a_k|^2$$
$$\|\sum_{k=-\infty}^\infty a_k f_k\|^2 = \sum_{k-\infty}^\infty |a_k|^2$$
by Parseval's formula. Hence S is an isometry and $\|S\| = 1$. S has no eigenvalues since if

$\lambda = 0$ then $a_k = 0$ all k

$\lambda \ne 0$ then $a_k = \lambda a_{k+1}$ all k and hence $\displaystyle\sum_{k=-\infty}^\infty |a_k|^2 = \infty$

unless $\sum_{k=-\infty}^\infty |a_k|^2 = 0$.

3. $(e_k)_{k=1}^{\infty}$ is a sequence in H, where $\|e_k\| = 1$ for all k. We want to show that

$$\sum_k |\langle x, e_k \rangle|^2 \leq \|x\|^2 (1 + (\sum_{k \neq \ell} |\langle e_k, e_\ell \rangle|^2)^{1/2}).$$

Note that

$$\sum_k |\langle x, e_k \rangle|^2 = \sum_k \langle x, e_k \rangle \overline{\langle x, e_k \rangle} = \langle x, \Sigma_k \langle x, e_k \rangle e_k \rangle \leq$$
$$\leq \|x\| \cdot \|\Sigma_k \langle x, e_k \rangle e_k\|.$$

Moreover

$$\begin{aligned}
\|\sum_k \langle x, e_k \rangle e_k\|^2 &= \sum_{k,\ell} \langle x, e_k \rangle \overline{\langle x, e_\ell \rangle} \langle e_k, e_\ell \rangle \leq \\
&\leq \sum_{k,\ell} |\langle x, e_k \rangle| |\langle x, e_\ell \rangle| |\langle e_k, e_\ell \rangle| = \\
&= \sum_k |\langle x, e_k \rangle|^2 + \sum_{k \neq \ell} (|\langle x, e_k \rangle| |\langle x, e_\ell \rangle|) |\langle e_k, e_\ell \rangle| \leq \\
&\leq \sum_k |\langle x, e_k \rangle|^2 + (\sum_{k \neq \ell} (|\langle x, e_k \rangle|^2 \cdot |\langle x, e_\ell \rangle|^2)^{1/2} \times \\
&\quad \times (\sum_{k \neq \ell} |\langle e_k, e_\ell \rangle|^2)^{1/2} \leq \\
&\leq \sum_k |\langle x, e_k \rangle|^2 + (\sum_{k,\ell} |\langle x, e_k \rangle|^2 |\langle x, e_\ell \rangle|^2)^{1/2} \times \\
&\quad \times (\sum_{k \neq \ell} |\langle e_k, e_\ell \rangle|^2)^{1/2} = \\
&= \sum_k |\langle e_k, e_\ell \rangle|^2 (1 + (\sum_{k \neq \ell} |\langle e_k, e_\ell \rangle|^2)^{1/2}).
\end{aligned}$$

This yields the result.

4. 5.86. Theory from book (and lectures).

Lösningsförslag till TMA401/MAN670 2004-05-29

1. Consider the BVP

$$\begin{cases} Lu = u''(x) - u(x) = -\frac{1}{2}(1 + u(x^2)), & x \in [0,1] \\ u(0) = u'(0) = 0,\ u \in C^2([0,1]) \end{cases} \qquad (*)$$

Calculation of Green's function

$$g(x,t) = (a_1(t)e^x + a_2(t)e^{-x})\theta(x-t) + b_1(t)e^x + b_2(t)e^{-x}$$

where

$$\begin{cases} a_1(t)e^t + a_2(t)e^{-t} = 0 \\ a_1(t)e^t - a_2(t)e^{-t} = 1 \end{cases} \quad \text{i.e.} \quad \begin{aligned} a_1(t) &= \tfrac{1}{2}e^{-t} \\ a_2(t) &= -\tfrac{1}{2}e^{t} \end{aligned}$$

and

$$\begin{cases} b_1(t) + b_2(t) = 0 \\ b_1(t) - b_2(t) = 0 \end{cases} \quad \text{i.e.} \quad \begin{aligned} b_1(t) &= 0 \\ b_2(t) &= 0 \end{aligned}$$

Hence we have $g(x,t) = \sinh(x-t)\theta(x-t)$.

Now set

$$T : C([0,1]) \longrightarrow C([0,1]),$$

where $Tu(x) = \int_0^1 g(x,t)(-\frac{1}{2}(1+u(t^2)))dt$. From Banach's fixed point theorem we conclude that $(*)$ has a unique solution if T is a contraction. For $u, v \in C([0,1])$ we have

$$\begin{aligned} |Tu(x) - Tv(x)| &= |\int_0^1 g(x,t)\frac{1}{2}(u(t^2) - v(t^2))dt| \le \\ &\le \frac{1}{2}\int_0^1 |g(x,t)|dt\|u-v\|_\infty = \frac{1}{2}(\cosh k - 1)\|u-v\|_\infty \le \\ &\le \underbrace{\frac{1}{4}(e + \frac{1}{e} - 2)}_{<1} \|u-v\|_\infty. \end{aligned}$$

Here T is a contraction and the statement follows.

2. Set, for $f \in L^2([a,b]), Tf(x) = \frac{1}{b-a}\int_a^b f(x)dx,\ x \in [a,b]$.

$Tf \in L^2([a,b])$ since

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_a^b (\frac{1}{b-a}|\int_a^b f(x)\,dx|)^2 dt = \\ &= (\frac{1}{b-a})^2 \int_a^b |\int_a^b f(x)dx|^2 dt \le \{\text{Hölder}\} \le \\ &\le (\frac{1}{b-a})^2 \int_a^b (b-a)\int_a^b |f(x)|^2 dxdt = \int_a^b |f(x)|^2 dt = \|f\|_{L^2}^2. \end{aligned}$$

T linear: easy to show.

T bounded: see above. In particular we get $\|T\| \le 1$.

To show that T is an orthogonal projection it suffices to show that $T^2 = T$ and $T^* = T$.

(a) Take $f \in L^2([a,b])$. Then

$$\begin{aligned} (T^2 f)(x) &= T(\frac{1}{b-a}\int_a^b f(t)dt) = \frac{1}{b-a}\int_a^b \frac{1}{b-a}\int_a^b f(x)dtds = \\ &= \frac{1}{b-a}\int_a^b f(t)dt = (Tf)(x), \quad \text{all} \quad x \in [a,b]. \end{aligned}$$

Hence $T^2 = T$.

(b) Take $f, g \in L^2([a,b])$. We obtain

$$\begin{aligned}\langle Tf, g\rangle &= \int_a^b \frac{1}{b-a}\int_a^b f(x)dx \cdot \overline{g(t)}dt = \\ &= \int_a^b f(x)\frac{1}{b-a}\int_a^b \overline{g(t)}dtdx = \\ &= \int_a^b f(x)\,\overline{\frac{1}{b-a}\int_a^b g(t)dt}dx = \langle f, Tg\rangle\end{aligned}$$

Hence $T^* = T$.

The statement is proved.

3. Let $h \in C([0,1]\times[0,1])$ be real-valued and

$$h(x,y) = h(y,x) > 0 \quad \text{all} \quad x, y \in [0,1]. \tag{$*$}$$

Set $Tf(x) = \int_0^1 h(x,y)f(y)dy$, $x \in [0,1]$, for $f \in L^2([0,1])$. We want to show that T has an eigenvalue $\lambda = \|T\|$ whitch is simple. (All eigenvalues λ satisfy $|\lambda| \leq \|T\|$). Since the kernel is continuous and satisfies $(*)$ we see that T is a compact, self-adjoint operator or $L^2([0,1])$ and hence has an eigenvalue $\lambda \in \mathbb{R}$ with $|\lambda| = \|T\|$. Since $h > 0$ we see that $\lambda = \|T\|$ (see first and second observation below). It remains to prove that this eigenvalue is simple.

First observation: f eigenfunction for $\lambda \Rightarrow f \in C([0,1])$ which follows from Lebesgues dominated convergence. Then
Second observation: f eigenfunction for $\lambda \Rightarrow f$ has constant sign, say $f \geq 0$, since if f changes sign, then

$$\lambda\|f\| = \|T\|\|f\| = \|Tf\| < \|T|f|\| \leq \|T\|\,\||f|\| = \|T\|\,\|f\|.$$

Moreover we can conclude that $f > 0$ since $h > 0$.

Third observation: f_1, f_2 eigenfunction for $\lambda \Rightarrow f_1 = \alpha f_2$ for some $\alpha \neq 0$.

To see this assume that it is false and set

$$s(\alpha) = |\{x \in [0,1] : f_1(x) - \alpha f_2(x) \geq 0\}|,\ \alpha \geq 0,$$

where $|E|$ denotes the measure of the set E. Here $s(0) = 1$, $s(\alpha)$ decreasing and $\lim_{a\to+\infty} s(\alpha) = 0$. Also $s(\alpha) = 1$ for $\alpha \in [0,\tilde{\alpha}]$ for some $\tilde{\alpha} > 0$ since $f_1, f_2 \in C([0,1])$ and $f_1, f_2 > 0$ on $[0,1]$. Then there exists $0 < \alpha_0 < \alpha_1$ such that $1 > s(\alpha_0) \geq s(\alpha_1) > 0$. [*)]

This means that

$$\begin{aligned} f_1(x) &< \alpha_0 f_1(x) &&\text{on a set of positive measure}\\ f_1(x) &\geq \alpha_0 f_2(x) &&\text{on a set of positive measure}\end{aligned}$$

but since also

$$f_1(x) \geq \alpha_1 f_2(x) \qquad \text{on a set of positive measure,}$$

together with $0 < \alpha_0 < \alpha_1$ and $f_2 > 0$ we see that

$$f_1(x) > \alpha_0 f_2(x) \qquad \text{on a set of positive measure.}$$

Then $f = f_1 - \alpha_0 f_2$ is an eigenfunction for λ (note that $f \neq 0$) that changes sign. Contradiciton!

The statement follows.

[*)] It cannot happen that $\exists\beta > 0$ sth.

$$\begin{cases} s(\alpha) = 1 & \text{for} \quad \alpha < \beta \\ s(\alpha) = 0 & \text{for} \quad \alpha > \beta\end{cases} \qquad \text{easy to see.}$$

and if there exists a $\beta > 0$ sth

$$\begin{aligned} s(\alpha) = 1 \quad &\text{for} \quad \alpha \leq \beta \\ s(\alpha) = 0 \quad &\text{for} \quad \alpha > \beta\end{aligned}$$

then $f_1 = \beta f_2$.

Lösningsskisser till tentamen i TMA401/MAN670, 2004-01-12

1. We will prove that
$$\begin{cases} u''(x)+u(x)=\frac{u(x)}{2+u^2(x)}, 0\le x\le \frac{\pi}{2} \\ u(0)=u(\frac{\pi}{2})=0, u\in C^2([0,\frac{\pi}{2}]) \end{cases}$$
has a unique solution.

   1. Determine the Green's function for $\begin{cases} u''+u=F \\ u(0)=m(\frac{\pi}{2})=0 \end{cases}$:
   Set $e(x,t)=a_1(t)\cos x+a_2(t)\sin x$, where $e(t,t)=0$ and $e'_x(t,t)=1$. This gives $e(x,t)=\sin(x-t)$. The Green's function takes the form
   $$g(x,t)=\sin(x-t)\theta(x-t)+b_1(t)\cos x+b_2(t)\sin x.$$
   Here $g(0,t)=g(\frac{\pi}{2},t)=0$ for $0<t<\frac{\pi}{2}$ implies that
   $$\begin{aligned} g(x,t)&=\sin(x-t)\theta(x-t)-\sin x\cos t= \\ &=\begin{cases} -\cos x\sin y, & x>t \\ -\sin x\cos t, & x<t \end{cases} \end{aligned}$$
   We see that $g(x,t)\le 0$ for all $x,t\in[0,\frac{\pi}{2}]$.
   2. Set
   $$\begin{cases} (Tu)(x)=\int_0^{\frac{\pi}{2}} g(x,t)\frac{u(t)}{2+u^2(t)}dt, & 0\le x\le\frac{\pi}{2} \\ u\in C([0,\frac{\pi}{2}]) \end{cases}$$
   The boundary value problem has a unique solution iff T has a unique fixed point.
   For $u,v\in C([0,\frac{\pi}{2}])$ we get
   $$\begin{aligned} |(Tu)(x)-(Tv)(x)| &\le \int_0^{\frac{\pi}{2}} |g(,t)||\frac{u(t)}{2+u^2(t)}-\frac{v(t)}{2+v^2(t)}|dt\le \\ &\le \{\text{mean value theorem}\}\le \\ &\le \frac{1}{2}\int_0^{\frac{\pi}{2}} |g(x,t)|dt\|u-v\|_\infty \le \frac{\pi}{4}\|u-v\|_\infty \end{aligned}$$
   This shows that T is a contraction on the space $C([0,\frac{\pi}{2}])$ and the conclusion follows from Banach's fixed point theorem.

2. Set $Tf(x)=\int_0^1(x+y)f(y)dy$ for $f\in L^2([0,1])$. T is bounded since
$$\begin{aligned} \|Tf\|_{L^2}^2 &= \int_0^1 |\int_0^1 (x+y)f(y)dy|^2dx \le \{\text{Hölder}\}\le \\ &\le \int_0^1\int_0^1 (x+y)^2dy\,\|f\|_{L^2}^2dx=\frac{7}{6}\|f\|_{L^2}^2 \end{aligned}$$
and hence $\|T\|\le\sqrt{\frac{7}{6}}$.
To calculate $\|T\|$ we observet hat T is a compact, self-adjoint operator on the Hilbert space $L^2([0,1])$ and hence
$$\|T\| = \sup_{\lambda \text{ eigenvalue to } T} |\lambda|.$$
Note that $Tf(x)=\int_0^1 f(y)dy\,x+\int_0^1 yf(y)dy$ is a polynomial of degree ≤ 1 and hence λ is an eigenvalue to T with eigenfunction $ax+b$ if
$$\lambda(ax+b)=\int_0^1(ay+b)dy\,x+\int_0^1 y(ay+b)dy,\ x\in[0.1].$$

i.e.,

$$\begin{cases} \lambda a = \frac{1}{2}a + b \\ \lambda b = \frac{1}{3}a + \frac{1}{2}b \end{cases}$$

This system has a non-trivial solution iff

$$\begin{vmatrix} \frac{1}{2} - \lambda & 1 \\ \frac{1}{3} & \frac{1}{2} - \lambda \end{vmatrix} = 0.$$

We obtain the eigenvalues $\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$ and so $\|T\| = \frac{1}{2} + \frac{1}{\sqrt{3}}$.

3. From problem 1 we can restate the problem as to show that the mapping,

$$Tu(x) = \lambda \int_0^{\frac{\pi}{2}} g(x,t) \frac{u(t)}{2 + u^2(t)} dt,\ 0 \leq x \leq \frac{\pi}{2},$$

for $u \in C([0, \frac{\pi}{2}])$, has a fixed point in $C([0, \frac{\pi}{2}])$ for all $\lambda \in \mathbb{R}$. For $|\lambda|$ large enough we cannot refer to the Banach's fixed point theorem since T is no longer a contraction. Instead we can use the Schander's fixed point theorem.

Fix a $\lambda \in \mathbb{R}$.

Note that $g(x,t)$ is a continuous function for $0 \leq x, t \leq \frac{\pi}{2}$ and $f(u) \equiv \lambda \frac{u}{2+u^2}$ is a bounded function for $u \in \mathbb{R}$ (note that λ is fixed.)

We will choose a closed convex set $S \subset C([0, \frac{\pi}{2}])$ such that the mapping $T : S \to S$ is continuous and the image set $T(S)$ is relatively compact in $C([0, \frac{\pi}{2}])$.
Take $S = \{u \in C([0, \frac{\pi}{2}]) : \|u\| \leq D\}$, where D is to be chosen such that $T(S) \subset S$. Since g is continuous on the compact set $\{(x,y) : 0 \leq x, t \leq \frac{\pi}{2}\}$ it is bounded on this set and so

$$\sup_{\substack{0 \leq x,t \leq \frac{\pi}{2} \\ u \in \mathbb{R}}} |g(x,t)f(u)| = E < \infty$$

which yields $|(Tu)(x)| \leq \frac{\pi}{2}E$ for all $x \in [0, \frac{\pi}{2}]$ and $u \in C([0, \frac{\pi}{2}])$. Hence $T(S) \subset S$ if $D = \frac{\pi}{2}E$.

Now it remains to prove that $T(S)$ is relatively compact in $C([0, \frac{\pi}{2}])$ and $TS \to S$ is continuous. The first statement follows from the Arzela-Aschi Theorem, since $T(S)$ is uniformly bounded and equicontinuous (the latter follows from the uniform continuity of g on $\{(x,t) : 0 \leq x, t \leq \frac{\pi}{2}\}$), and the second statement follows from the uniform continuity of $f(y)$ for $-D \leq u \leq D$ and the boundedness of g.

The existence of a fixed point for T now follows from Schander's fixed point theorem.

(Remark: As a matter of fact one trivially observes that $u = 0$ is a solution to the problem. However to treat the problem with the RHS in the differential equation replaced by the original one $+1$ is harder but the method above yields a solution.)

4. See textbook.

5. $T : X \to X$, where X is a real normal space, is continuous and satisfies

$$T(x + y) = T(x) + T(y) \quad \text{for all} \quad x, y \in X. \tag{$*$}$$

We shall prove that

$$T(\lambda x) = \lambda T(x) \quad \text{for all} \quad x \in X \text{ and } \lambda \in \mathbb{R} \tag{$**$}$$

It follows from $(*)$ that

- $T(\mathbf{0}) = \mathbf{0}$ since $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = 2T(\mathbf{0})$.
- $T(nx) = nT(x)$ for positive integrs n since

$$T(nx) = T(x + (n-1)x) = T(x) + T((n-1)x) = \ldots = nT(x)$$

- $T(\frac{1}{n}x) = \frac{1}{n}T(x)$ for positive integers n since

$$T(x) = T(n(\frac{1}{n}x)) = nT(\frac{1}{n}x).$$

Combining these observations we see that $T(\lambda x) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{Q}$. To prove (**) we fix a $\lambda \in \mathbb{R}$ and an $x \in X$ and take a sequence $\lambda_n \in \mathbb{Q}$, $n = 1, 2, \ldots$, such that $\lambda_n \to \lambda$ in $\mathbb{R}$. This implies that $\lambda_n x \to \lambda x$ in X. It also implies $\lambda_n T(x) \to \lambda T(x)$ in X. Since T is continuous we conclude that $T(\lambda_n x) \to T(\lambda x)$ in X. But, $T(\lambda_n x) = \lambda_n T(x) \longrightarrow \lambda T(x)$ in X and so

$$T(\lambda x) = \lambda T(x)$$

The statement is proved.

6. Let $(x_n)_{n=1}^{\infty}$ be an ON-basis in H and $(y_n)_{n=1}^{\infty}$ an ON-sequence in H such that $\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1$. We shall show that also $(y_n)_{n=1}^{\infty}$ is an ON-basis.

   Set $S = \overline{\text{span}\{y_n : n = 1, 2, \ldots\}}$. Then S is a closed subspace of H. It remains to prove that $S = H$.

   Assume that $S \neq H$. Then $S^{\perp} \neq \{\mathbf{0}\}$ and there is an $x \in S^{\perp}$ with $\|x\| > 0$. We have

$$\langle x, y_n \rangle = 0, n = 1, 2, \ldots$$

   since $x \in S^{\perp}$. Parseval's formula yields

$$\begin{aligned} \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle - \langle x, y_n \rangle|^2 = \\ \sum_{n=1}^{\infty} |\langle x, x_n - y_n \rangle|^2 \leq \{\text{Cauchy-Schwartz}\} \leq \\ \leq \sum_{n=1}^{\infty} \|x\|^2 \|x_n - y_n\|^2 < \|x\|^2. \end{aligned}$$

   This yields a contradiciton and hence $S = H$.

Kortfattade lösningsskisser till tentamen i TMA401/MAN670, 2003-08-30

Problem 1
We will prove that

$$\begin{cases} u''(x) - u(x) = \lambda \frac{u(x)}{1+u^2(x)}, & x \in (0,1) \\ u(0) - u'(0) + u(1) = u(0) + u'(0) + 2u'(1) = 0 \\ u \in C^2 \end{cases}$$

has a unique solution for $|\lambda|$ small enough.

1. Determine the Green's function $g(x,t)$ for
$\begin{cases} u'' - u = F & x \in (0,1) \\ u(0) - u'(0) + u(1) = u(0) + u'(0) + 2u'(1) = 0 \end{cases}$
Set .... (standard calculation) ....

2. Set
$$\begin{cases} (Tu)(x) = \displaystyle\int_0^1 g(x,t) \cdot \lambda \frac{u(t)}{1+u^2(t)}\, dt, & 0 \le x \le 1 \\ u \in C[0,1] \end{cases}$$
The boundary value problem has a unique solution iff T has a unique fixed point. For $u, v \in C[0,1]$ we obtain
$$|(Tu)(x) - (Tv)(x)| \le \text{......(standard calculations).....} \le C(\lambda)\|u-v\|_\infty.$$
This shows that T is a contraction on the Banach space $C[0,1]$ provided $C(\lambda) < 1$ and the conclusion follows from Banach's fixed point theorem.

Problem 2
The solution is a straight forward application of the Gram-Schmidt process and we only refer to the textbook for more information.
Problem 3
We have to show that the ON-sequence $(u_n)_{n=1}^\infty$ in $L^2([0,1])$ is complete if

$$\sum_{n=1}^\infty \left| \int_0^x u_n(t)dt \right|^2 = x \text{ for all } x \in [0,1] \tag{1}$$

Formula (1) can be reformulated as

$$\sum_{n=1}^\infty |\langle \chi_{[0,x]}, u_n\rangle|^2 = \|\chi_{[0,x]}\|^2 \text{ for all } x \in [0,1],$$

where

$$\chi_I(t) = \begin{cases} 1 & t \in I \\ 0 & t \notin I \end{cases},$$

since

$$\langle \chi_{[0,x]}, u_n\rangle = \int_0^1 \chi_{[0,x]}(t)\overline{u_n(t)}dt = \overline{\int_0^x u_n(t)dt}$$

and

$$\|\chi_{[0,x]}\|^2 = \int_0^1 |\chi_{[0,x]}(t)|^2 dt = \int_0^x dt = x.$$

To show that $(u_n)_{n=1}^\infty$ is complete it is enough to show that

$$\sum_{n=1}^\infty |\langle f, u_n\rangle|^2 = \|f\|^2 \text{ for all } f \in L^2([0,1]). \tag{2}$$

Here it suffices to show that (2) is true for a dense set A in $L^2([0,1])$, since if $f_k \to f$ in L^2 and (2) is true for $f_k, k = 1, 2, \ldots$, then we have

$$\begin{aligned}
&0 \le \|f\|^2 - \sum_{n=1}^{\infty} |\langle f, u_n\rangle|^2 = \\
&= \|(f - f_k) + f_k\|^2 - \sum_{n=1}^{\infty} |\langle (f - f_k) + f_k, u_n\rangle|^2 \le \\
&\le \|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\| + \|f_k\|^2 - \\
&- \sum_{n=1}^{\infty} (\langle f - f_k, u_n\rangle + \langle f_k, u_n\rangle)\overline{(\langle f - f_k, u_n\rangle + \langle f_k, u_n\rangle)} = \\
&= \|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\|^2 + \sum_{n=1}^{\infty} |\langle f_k, u_n\rangle|^2 - \\
&\quad - \sum_{n=1}^{\infty} (|\langle f - f_k, u_n\rangle|^2 + 2\mathrm{Re}\langle f - f_k, u_n\rangle \overline{\langle f_k, u_n\rangle} + \\
&\quad + |\langle f_k, u_n\rangle|^2) \le \\
&\quad \le \|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\| + \|f - f_k\|^2 + \\
&\quad + \sum_{n=1}^{\infty} |\langle f - f_k, u_n\rangle| \cdot |\langle f_k, u_n\rangle| \le \\
&\quad \le 2\|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\| + \\
&\quad + (\sum_{n=1}^{\infty} |\langle f - f_k, u_n\rangle|^2)^{1/2} (\sum_{n=1}^{\infty} |\langle f_k, u_n\rangle|^2) \le \\
&\quad \le 2\|f - f_k\|^2 + 2\|f_k\| \cdot \|f - f_k\| + \|f - f_k\| \|f_k\| \to 0
\end{aligned}$$

as $k \to \infty$. Here we have used that $\sup_k \|f_k\| < \infty$, which follows from $f_k \to f$ in L^2, and repeatedly applied Bessel's inequality.
Now we take A to be the set of all finite linear combinations of χ_I, where the I:s are subintervals of $[0,1]$. Then A is dense in $L^2([0,1])$. Set $g = \sum_{k=1}^{N} a_k \chi_{I_k}$ where the intervals I_k are pairwise disjoint subintervals of $[0,1]$ and $a_k \in \mathbb{C}$.
We know that (2) is valid for $f = \chi_{[0,x]}$. It remains to show that:

- $f = \chi_{[y,x]}$ satisfies (2)
- f, g satisfies (2) and have disjoint support implies that any linear combination of f, g satisfies (2).

Fix $0 < y < x \le 1$. We note that

$$\|\chi_{[0,x]}\|^2 = \sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n\rangle|^2$$

$$\|\chi_{[0,y]}\|^2 = \sum_{n=1}^{\infty} |\langle \chi_{[0,y]}, u_n\rangle|^2. \tag{3}$$

This yields

$$\|\chi_{[y,x]}\|^2 = x - y$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} |\langle \chi_{[y,x]}, u_n\rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]} - \chi_{[0,y]}, u_n\rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n\rangle - \langle \chi_{[0,y]}, u_n\rangle|^2 = \\
&= \sum_{n=1}^{\infty} |\langle \chi_{[0,x]}, u_n\rangle|^2 + \sum_{n=1}^{\infty} |\langle \chi_{[0,y]}, u_n\rangle|^2 - \\
&- 2\mathrm{Re} \sum_{n=1}^{\infty} \langle \chi_{[0,x]}, u_n\rangle \overline{\langle \chi_{[0,y]}, u_n\rangle} = \\
&= x + y - 2\mathrm{Re}\langle \chi_{[0,x]}, \sum_{n=1}^{\infty} \langle \chi_{[0,y]}, u_n\rangle u_n\rangle = (!) = \\
&= x + y - 2\mathrm{Re}\langle \chi_{[0,x]}, \chi_{[0,y]}\rangle = x + y - 2\|\chi_{[0,y]}\|^2 = x - y.
\end{aligned}$$

Note that at (!) we have used the fact

$$\chi_{[0,y]} = \sum_{n=1}^{\infty} \langle \chi_{[0,y]}, u_n\rangle u_n$$

which follows from (3).
We have thus found that $\chi_{[y,x]}$ satisfies (2). To prove the second statement we have to do similar calculations (do it yourself!!!) and the full statement that $g \in A$ implies g satisfies (2) follows by induction over N (see the expression for g above).

Problem 4 & 5 & 6

See the textbook

Förslag till lösningar till tentamen i TMA401/MAN670, 2003-05-31

Problem 1
We will prove that

$$\begin{cases} u''(x) + u'(x) = \arctan u(x^2), & x \in (0,1) \\ u(0) = u(1) = 0 \\ u \in C^2 \end{cases}$$

has a unique solution.

1. Determine the Green's function for $\begin{cases} u'' + u' = F & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$:

   Set $e(x,t) = a_1(t) + a_2(t)e^{-x}$ where $\begin{cases} e(t,t) = 0 \\ e'_x(t,t) = 1 \end{cases}$. This gives $e(x,t) = 1 - e^{t-x}$. The Green's function takes the form

   $$g(x,t) = \theta(x-t)(1 - e^{t-x}) + b_1(t) + b_2(t)e^{-x}.$$

   Here $g(0,t) = g(1,t) = 0$, $0 < t < 1$ implies

   $$\begin{cases} b_1(t) = \frac{e^t - 1}{e-1} \\ b_2(t) = -\frac{e^t-1}{e-1}, \end{cases}$$

   Hence $g(x,t) = \theta(x-t)(1 - e^{t-x}) + \frac{e^t-1}{e-1}(1 - e^{-x})$. We see (a simple calculation) that $g(x,t) \leq 0$ for all x, t.

2. Set

   $$\begin{cases} (Tu)(x) = \int_0^1 g(x,t) \arctan u(t^2)\, dt, & 0 \leq x \leq 1 \\ u \in C[0,1] \end{cases}$$

   The boundary value problem has a unique solution iff T has a unique fixed point. For $u, v \in C[0,1]$ we obtain

   $$\begin{aligned} |(Tu)(x) - (Tv)(x)| &= \int_0^1 |g(x,t)||\arctan u(t^2) - \arctan v(t^2)|\, dt \leq \\ &\leq \{\text{mean value theorem}\} \leq \\ &\leq \int_0^1 |g(x,t)||u(t^2) - v(t^2)|\, dt \leq \\ &\leq \int_0^1 (-g(x,t))\, dt \|u - v\|_\infty. \end{aligned}$$

   Vi note that (small calculation)

   $$\int_0^1 (-g(x,t)\, dt = \max_{0 \leq x \leq 1} (-x + \frac{e}{e-1}(1 - e^{-x}) \equiv c < 1$$

   and hence

   $$\|Tu - Tv\|_\infty \leq c\|u - v\|_\infty.$$

   This shows that T is a contraction on the Banach space $C[0,1]$ and the conclusion follows from Banach's fixed point theorem.

Problem 2
Let T, H, $(e_n)_{n=1}^\infty$ and $(f_n)_{n=1}^\infty$ be as in the formulation of the problem. We note that since $(e_n)_{n=1}^\infty$ is an ON-basis for H we have

$$f_n = \Sigma_{k=1}^\infty \langle f_n, e_k \rangle e_k, \quad n = 1, 2, \ldots$$

and since T is continuous we get

$$Tf_n = \Sigma_{k=1}^{\infty}\langle f_n, e_k\rangle Te_k, \quad n = 1, 2, \dots$$

This yields

$$\Sigma_{n=1}^{\infty}\|Tf_n\|^2 = \Sigma_{n=1}^{\infty}\Sigma_{k=1}^{\infty}\Sigma_{l=1}^{\infty}\langle f_n, e_k\rangle\langle Te_k, Te_l\rangle\overline{\langle f_n, e_l\rangle}.$$

and since the series is absolutely convergent we can change the order of summation. Observing that

$$\Sigma_{n=1}^{\infty}\langle f_n, e_k\rangle\overline{\langle f_n, e_l\rangle} = \langle \Sigma_{n=1}^{\infty}\langle e_l, f_n\rangle f_n, e_k\rangle = \langle e_l, e_k\rangle$$

we have

$$\Sigma_{n=1}^{\infty}\|Tf_n\|^2 = \Sigma_{k=1}^{\infty}\|Te_k\|^2.$$

Moreover $x = \Sigma_{n=1}^{\infty}\langle x, e_n\rangle e_n$ implies $Tx = \Sigma_{n=1}^{\infty}\langle x, e_n\rangle Te_n$ and

$$\|Tx\| = \|\Sigma_{n=1}^{\infty}\langle x, e_n\rangle Te_n\| \leq \Sigma_{n=1}^{\infty}|\langle x, e_n\rangle|\|Te_n\| \leq$$

$$\leq (\Sigma_{n=1}^{\infty}|\langle x, e_n\rangle|^2)^{\frac{1}{2}}(\Sigma_{n=1}^{\infty}\|Te_n\|^2)^{\frac{1}{2}} = \|x\|(\Sigma_{n=1}^{\infty}\|Te_n\|^2)^{\frac{1}{2}}$$

by Parseval's formula. Hence we get

$$\|T\|^2 \leq \Sigma_{n=1}^{\infty}\|Te_n\|^2.$$

Problem 3 Let

$$Mf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0$$

for $f \in L^2(\mathbb{R}_+)$. We observe that $Mf(x) \in \mathbb{R}$ for $x > 0$. Moreover for every continuous function f with compact support in $\{x \in \mathbb{R} : x > 0\}$ we note that

$$\int_0^{\infty}\frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2\,dt < \infty$$

and that

$$\|Mf\|^2 \leq \int_0^{\infty}\frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2\,dt = [-\frac{1}{t}(\int_0^t |f(s)|\,ds)^2]_0^{\infty} + 2\int_0^{\infty}\frac{1}{t}\int_0^t |f(s)|\,ds\,f(t)\,dt \leq$$

$$\leq 0 + 2(\int_0^{\infty}\frac{1}{t^2}(\int_0^t |f(s)|\,ds)^2)^{\frac{1}{2}}(\int_0^{\infty}|f(s)|^2\,ds)^{\frac{1}{2}},$$

i.e. we get

$$\|Mf\| \leq 2\|f\|.$$

Now we recall that the set of continuous functions with compact support in $\{x \in \mathbb{R} : x > 0\}$ is dense in $L^2(\mathbb{R}_+)$ and from the inequality above we get that $Mf \in L^2$ for every $f \in L^2$ and that $\|M\| \leq 2$, and in particular that M is bounded. A straight-forward calculation show that

$$M^*f(x) = \int_x^{\infty}\frac{1}{t}f(t)\,dt, \quad x > 0$$

and that

$$\|(I - M)f\|^2 = \|f\|^2$$

for all $f \in L^2$. From this it follows that

$$\|I - M\| = 1.$$

Problem 4 & 5
See the textbook
Problem 6 We want to show that there exists a $C > 0$ such that for every $y \in \mathcal{R}(I + T)$ there exists a $x \in X$ with $(I + T)x = y$ satisfying

$$\|x\| \leq C\|y\|.$$

First we fix a $y \in \mathcal{R}(I+T)$ and set

$$d(y) = \inf\{\|x\| : (I+T)x = y\}.$$

Claim: There exists an $\tilde{x} \in X$ with $(I+T)\tilde{x} = y$ such that $\|\tilde{x}\| = d(y)$. To see this take a sequence $(x_n)_{n=1}^{\infty}$ with $(I+T)x_n = y$ such that $\|x_n\| \to d(y)$. Since this sequence is bounded there is a converging subsequence of $(Tx_n)_{n=1}^{\infty}$, let this still be denoted by $(Tx_n)_{n=1}^{\infty}$, converging to say $z \in X$. Here we used the fact that T is compact. But then $x_n \to y - z$ in X and hence $\tilde{x} = y - z$ has the desired property.
Now assume that there is no $C > 0$ with the property above. Then there are sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ satisfying $(I+T)x_n = y_n$ such that

$$\frac{\|\tilde{x}_n\|}{\|y_n\|} \to \infty.$$

Since T is linear we can without loss of generality assume that $\|\tilde{x}_n\| = 1$ for all n. Since T is compact there exists a converging subsequence of $(T\tilde{x}_n)_{n=1}^{\infty}$, call it still $(T\tilde{x}_n)_{n=1}^{\infty}$, converging to say v in X. We also have

$$\tilde{x}_n \to -v \quad \text{in } X.$$

But since $\|y_n\| \to 0$ in X ($\|\tilde{x}_n\| = 1$ for all n) and since T is continuous (T is compact linear) we obtain $v = -Tv$. By the definition of $\tilde{x}_n$ this yields a contradiction since $(I+T)(\tilde{x}_n - v) = y_n$ and

$$\|\tilde{x}_n - v\| \geq \|\tilde{x}_n\| = 1$$

is valid for every n. The conclusion in problem 6 follows.

Förslag till lösningar till tentamen i TMA401/MAN670, 2002-08-21

Uppgift 1
Givet

$$Af(x) = \int_0^1 (x-y)f(y)\,dy, \quad 0 \le x \le 1,$$

dvs. A är en integraloperator på Hilbertrummet $L^2([0,1])$ med kärnan $k(x,y) = x - y$. Den adjungerade operatorn A^* är då också en integraloperator men med kärnan $k^*(x,y) = \overline{y-x} = y - x$. Vi får för $f \in L^2([0,1])$ att

$$A^*Af(x) = \int_0^1 (y-x)Af(y)\,dy = \int_0^1 (y-x)\Big(\int_0^1 (y-z)f(z)\,dz\Big)\,dy =$$

$$= \int_0^1 \Big(\int_0^1 (y-x)(y-z)\,dy\Big)f(z)\,dz = \int_0^1 \Big(\frac{1}{3} - \frac{1}{2}(x+z) + xz\Big)f(z)\,dz,$$

där vi använt Fubinis sats.
För att beräkna $\|A\|$ noterar vi att A^*A är en självadjungerad kompakt operator och $\|A\| = \sqrt{\|A^*A\|}$. Vidare gäller för en självadjungerad kompakt operator att dess norm är lika med det största reella tal som är absolutbeloppet av ett egenvärde till operatorn ifråga. Vi noterar att $A^*Af(x) = a(f)x + b(f)$ där $a(f), b(f)$ är reella tal som beror på $f \in L^2([0,1])$. Följdaktligen har egenfunktioner till A^*A formen $e(x) = ax + b$ varför vi ansätter

$$A^*Ae(x) = \lambda e(x), \quad e(x) = ax + b.$$

En liten kalkyl ger

$$\frac{a}{12}x + \frac{b}{12} = \lambda(ax+b), \quad \text{alla } x \in [0,1],$$

dvs det enda egenvärdet $\lambda \neq 0$ är $\frac{1}{12}$. Vi har alltså $\|A\| = \sqrt{\frac{1}{12}}$.
Uppgift 2
Vi ska visa att

$$\begin{cases} -u''(x) = 2 + \frac{1}{1+u^2(x)}, & x \in (0,1) \\ u(0) = u(1) = 0 \\ u \in C^2 \end{cases}$$

har en entydigt bestämd lösning.

1. Greenfunktioen till $\begin{cases} u'' = F & x \in (0,1) \\ u(0) = u(t) = 0 \end{cases}$:

   Antag $e(x,t) = a_1(t) + a_2(t)e^{-x}$ uppfyller $\begin{cases} e(t,t) = 0 \\ e'_x(t,t) = 1 \end{cases}$. Detta ger $e(x,t) = x - t$. Greenfunktionen ges av

   $$g(x,t) = \theta(x-t)(x-t) + b_1(t) + b_2(t) - x.$$

   Villkoren $g(0,t) = g(1,t)$, $0 < t < 1$ ger

   $$\begin{cases} b_1(t) = 0 \\ b_2(t) = t - 1,\ 0 < t < 1. \end{cases}$$

   Alltså $g(x,t) = \theta(x-t)(x-t) + (t-1)x$. Vi noterar att $g(x,t) \le 0$ alla x, t.

2. Sätt

   $$\begin{cases} (Tu)(x) = -\displaystyle\int_0^1 g(x,t)\Big(2 + \frac{1}{1+u^2(x)}\Big)\,dt, & 0 \le x \le 1 \\ u \in C[0,1] \end{cases}$$

   Det ursprungliga problemet har en unik lösning omm T har en unik fixpunkt.

För $u, v \in C[0,1]$ gäller

$$|(Tu)(x) - (Tv)(x)| = \int_0^1 |g(x,t)||\frac{1}{1+u^2(t)} - \frac{1}{1+v^2(t)}|\,dt \leq$$
$$\leq \int_0^1 |g(x,t)||\frac{(u(t)+v(t))(u(t)-v(t))}{(1+u^2(t))(1+v^2(t))}|\,dt \leq$$
$$\leq \int_0^1 |g(x,t)||\frac{u(t)+v(t)}{(1+u^2(t))(1+v^2(t))}|\,dt\|u-v\|_\infty.$$

Vi noterar att $|\frac{a+b}{(1+a^2)(1+b^2)}| \leq \frac{1}{2}|\frac{2a}{1+a^2}| + \frac{1}{2}|\frac{2b}{1+b^2}| \leq 1$ för alla reella tal a, b samt att

$$\int_0^1 |g(x,t)|\,dt \leq \max_{0\leq x\leq 1} \frac{x}{2}(1-x) = \frac{1}{8}$$

vilket ger

$$\|Tu - Tv\|_\infty \leq \frac{1}{8}\|u-v\|_\infty.$$

Detta visar att T är en kontraktion på Banachrummet $C[0,1]$ och påståendet följer från Banach fixpunktssats.

Uppgift 3
Hilbert-Schmidts sats ger

$$0 \leq \langle Tx, x\rangle = \Sigma_i \lambda_i |\langle x, e_i\rangle|^2$$

där $\lambda_i \geq 0$, e_i, $i = 1, 2, \ldots$, betecknar egenvärdena respektive motsvarande normerade egenvektorer till operatorn T. Låt n vara ett fixerat positivt heltal > 1 (om $n = 1$ är påståendet trivialt sant). Hölders olikhet med exponenterna n och n^*, där $1 = \frac{1}{n} + \frac{1}{n^*}$, tillsammans med Hilbert-Schmidts sats ger

$$0 \leq \langle Tx, x\rangle \leq (\Sigma_i \lambda_i^n |\langle x, e_i\rangle|^{\frac{2}{n}n})^{1/n} \cdot (\Sigma_i |\langle x, e_i\rangle|^{(2-\frac{2}{n})n^*})^{1/n^*} = \langle T^n x, x\rangle^{1/n} \cdot \|x\|^{2(n-1)/n}.$$

Uppgift 4 & 5
Se kursboken.

Förslag till lösningar till tentamen i TMA401/MAN670, 2002-06-01

Uppgift 1
Givet

$$Af(x) = \int_0^1 (x-y)f(y)\,dy, \quad 0 \le x \le 1,$$

dvs. A är en integraloperator på Hilbertrummet $L^2([0,1])$ med kärnan $k(x,y) = x - y$. Den adjungerade operatorn A^* är då också en integraloperator men med kärnan $k^*(x,y) = \overline{y-x} = y - x$. Vi får för $f \in L^2([0,1])$ att

$$A^*Af(x) = \int_0^1 (y-x)Af(y)\,dy = \int_0^1 (y-x)(\int_0^1 (y-z)f(z)\,dz)\,dy =$$

$$= \int_0^1 (\int_0^1 (y-x)(y-z)\,dy)f(z)\,dz = \int_0^1 (\frac{1}{3} - \frac{1}{2}(x+z) + xz)f(z)\,dz,$$

där vi använt Fubinis sats.
För att beräkna $\|A\|$ noterar vi att A^*A är en självadjungerad kompakt operator och $\|A\| = \sqrt{\|A^*A\|}$. Vidare gäller för en självadjungerad kompakt operator att dess norm är lika med det största reella tal som är absolutbeloppet av ett egenvärde till operatorn ifråga. Vi noterar att $A^*Af(x) = a(f)x + b(f)$ där $a(f), b(f)$ är reella tal som beror på $f \in L^2([0,1])$. Följdaktligen har egenfunktioner till A^*A formen $e(x) = ax + b$ varför vi ansätter

$$A^*Ae(x) = \lambda e(x), \quad e(x) = ax + b.$$

En liten kalkyl ger

$$\frac{a}{12}x + \frac{b}{12} = \lambda(ax+b), \quad \text{alla } x \in [0,1],$$

dvs det enda egenvärdet $\lambda \neq 0$ är $\frac{1}{12}$. Vi har alltså $\|A\| = \sqrt{\frac{1}{12}}$.

Uppgift 2
Vi ska visa att

$$\begin{cases} -u''(x) = 2 + \frac{1}{1+u^2(x)}, & x \in (0,1) \\ u(0) = u(1) = 0 \\ u \in C^2 \end{cases}$$

har en entydigt bestämd lösning.

1. Greenfunktioen till $\begin{cases} u'' = F & x \in (0,1) \\ u(0) = u(t) = 0 \end{cases}$:

   Antag $e(x,t) = a_1(t) + a_2(t)e^{-x}$ uppfyller $\begin{cases} e(t,t) = 0 \\ e_x'(t,t) = 1 \end{cases}$. Detta ger $e(x,t) = x - t$. Greenfunktionen ges av

   $$g(x,t) = \theta(x-t)(x-t) + b_1(t) + b_2(t) - x.$$

   Villkoren $g(0,t) = g(1,t)$, $0 < t < 1$ ger

   $$\begin{cases} b_1(t) = 0 \\ b_2(t) = t - 1,\ 0 < t < 1. \end{cases}$$

   Alltså $g(x,t) = \theta(x-t)(x-t) + (t-1)x$. Vi noterar att $g(x,t) \le 0$ alla x, t.

2. Sätt

   $$\begin{cases} (Tu)(x) = -\int_0^1 g(x,t)(2 + \frac{1}{1+u^2(x)})\,dt, & 0 \le x \le 1 \\ u \in C[0,1] \end{cases}$$

   Det ursprungliga problemet har en unik lösning omm T har en unik fixpunkt.

För $u, v \in C[0,1]$ gäller

$$|(Tu)(x) - (Tv)(x)| = \int_0^1 |g(x,t)|| \frac{1}{1+u^2(t)} - \frac{1}{1+v^2(t)}| \, dt \leq$$
$$\leq \int_0^1 |g(x,t)|| \frac{(u(t)+v(t))(u(t)-v(t))}{(1+u^2(t))(1+v^2(t))}| \, dt \leq$$
$$\leq \int_0^1 |g(x,t)|| \frac{u(t)+v(t)}{(1+u^2(t))(1+v^2(t))}| \, dt \|u-v\|_\infty.$$

Vi noterar att $|\frac{a+b}{(1+a^2)(1+b^2)}| \leq \frac{1}{2}|\frac{2a}{1+a^2}| + \frac{1}{2}|\frac{2b}{1+b^2}| \leq 1$ för alla reella tal a, b samt att

$$\int_0^1 |g(x,t)| \, dt \leq \max_{0 \leq x \leq 1} \frac{x}{2}(1-x) = \frac{1}{8}$$

vilket ger

$$\|Tu - Tv\|_\infty \leq \frac{1}{8}\|u-v\|_\infty.$$

Detta visar att T är en kontraktion på Banachrummet $C[0,1]$ och påståendet följer från Banach fixpunktssats.

Uppgift 3

Låt T vara en avbildning på ett normerat rum X som uppfyller följande villkor: Det finns ett reellt tal C och ett reellt tal $\alpha > 1$ sådana att

$$\|T(x) - T(y)\| \leq C\|x-y\|^\alpha, \text{ alla } x, y \in X.$$

Vi ska visa att $T(x) = T(\mathbb{0})$ för alla $x \in X$.
Fixera ett godtyckligt $x \in X$. Sätt $\delta = \|T(x) - T(\mathbb{0})\|$. Fixera ett positivt heltal n och sätt $x_k = \frac{k}{n}x \in X$ för $k = 0, 1, 2, \ldots, n$. Då gäller

$$\delta = \|T(x_n) - T(x_{n-1}) + T(x_{n-1}) - \ldots - T(x_0)\| \leq$$

$$\leq \Sigma_{k=0}^{n-1}\|T(x_{k+1}) - T(x_k)\| \leq C\Sigma_{k=0}^{n-1}\|x_{k+1} - x_k\|^\alpha =$$

$$= C\|x\|^\alpha \cdot n^{\alpha-1} \to 0, \quad n \to \infty.$$

Detta medför att $\delta = 0$ och påstendet i uppgiften är visat.
Anm: Om $T : \mathbf{R} \to \mathbf{R}$ uppfyller

$$|T(x) - T(y)| \leq C|x-y|^\alpha, \text{ alla } x, y \in \mathbf{R}$$

så gäller att

$$\lim_{0 \neq h \to 0} |\frac{T(x+h) - T(x)}{h} - 0| = 0, \text{ alla } x \in \mathbf{R}$$

dvs. T är en deriverbar funktion med derivatan $= 0$ för varje $x \in \mathbf{R}$ och alltså är T en konstant funktion.

Uppgift 4 & 5

Se kursboken.

Uppgift 6

Låt T vara en självadjungerad operator på ett Hilbertrum H för vilken T^n är kompakt för något heltal $n \geq 2$. Vi ska visa att T är kompakt.
Vi noterar att T självadjungerad innebär (per definition) att T är en begränsad operator. T är kompakt om vi kan visa att T^k kompakt implicerar att T^{k-1} är kompakt för godtyckligt heltal $k \geq 2$.
Antag nu att T^k är kompakt för fixt $k \geq 2$. Låt $(x_n)_{n=1}^\infty$ vara en begränsad följd i H, dvs det finns ett reellt tal M sådant att $\|x_n\| \leq M$ för alla n. Då T^k är kompakt finns det en delföljd (x_{p_n}) av

(x_n) för vilken $(T^k x_{p_n})$ konvergerar i H. Då konvergerar också $(T^{k-1}x_{p_n})$ i H eftersom

$$\begin{aligned}
\|T^{k-1}x_{p_n} - T^{k-1}x_{p_m}\|^2 = \langle T^{k-1}(x_{p_n} - x_{p_m}), T^{k-1}(x_{p_n} - x_{p_m})\rangle &= \\
= \langle T^{k-2}(x_{p_n} - x_{p_m}), T^{k}(x_{p_n} - x_{p_m})\rangle &\leq \\
\leq \|T^{k-2}(x_{p_n} - x_{p_m})\| \cdot \|T^k x_{p_n} - T^k x_{p_m}\| &\leq \\
\leq \|T^{k-2}\| \cdot \|x_{p_n} - x_{p_m}\| \cdot \|T^k x_{p_n} - T^k p_{p_m}\| &\leq \\
\leq \underbrace{2\|T\|^{k-2}M}_{<\infty} \cdot \underbrace{\|T^k x_{p_n} - T^k x_{p_m})\|}_{\to 0,\ n,m\to\infty}&
\end{aligned}$$

och varje Cauchyföljd i ett Hilbertrum konvergerar. Detta medför att T^{k-1} är kompakt och påståendet i uppgiften är visat.

Förslag till lösningar till tentamen i TMA400, 2001-05-30

1. A är integraloperator med kärnan $a(x,y) = e^{x+y}\cos(x+y)$, där $a \in C([0,\pi]\times[0,\pi])$ och $a(x,y) = \overline{a(y,x)}$.

   a) Banachrummet $C[0,\pi]$: För $u \in C[0,\pi]$ gäller
$$|Au(x)| \leq \int_0^\pi e^{x+y}|\cos(x+y)|dy\|u\|_\infty \equiv I(x)\|u\|_\infty, x \in [0,\pi],$$
   där $I \in C[0,\pi]$. Alltså $\|A\| \leq \max_{x\in[0,\pi]} I(x) = I(x_0)$ för något $x_0 \in [0,\pi]$. Vidare gäller $I(x_0) = \lim\limits_{n\to\infty} Au_n(x)$ där
$$u_n(x) = \begin{cases} +1 \text{ då } \min(|x+x_0-\frac{\pi}{2}|, |x+x_0-\frac{3\pi}{2}|) > \frac{1}{n} \;\&\; \cos(x+x_0) > 0 \\ -1 \text{ då } \min(|x+x_0-\frac{\pi}{2}|, |x+x_0-\frac{3\pi}{2}|) > \frac{1}{n} \;\&\; \cos(x+x_0) < 0 \\ \text{till beloppet} \leq 1 \text{ och kontinuerlig för övrigt} \end{cases}$$
   Här approximerar u_n-funktionerna funktion $\text{sign}(\cos(x_0+\cdot))$. (Liten) kalkyl ger $I(x_0) = I(\frac{\pi}{2}) = \frac{1}{2}(e^{\frac{3\pi}{2}} + e^{\frac{\pi}{2}})$.

   b) Banachrummet $L^2[0,\pi]$: Då L^2 är Hilbertrum och A är självadjungerad gäller
$$\|A\| = \sup\{|\lambda| : \lambda \text{ genvärde till } A\}.$$
   Då $Au(x) = \int_0^\pi e^x \cdot e^y(\cos x\cos y - \sin x \sin y)u(y)dy = ae^x\cos x + be^x \sin x$ fås egenvärdena λ som lösningar till $A(ae^x\cos x + be^x\sin x) = \lambda(ae^x\cos x + be^x \sin x)$ för $|a|+|b| > 0$, dvs
$$\begin{cases} \left(\frac{3}{8} - \frac{\lambda}{e^{2\pi}-1}\right)a - \frac{1}{8}b = 0 \\ \frac{1}{8}a - \left(\frac{3}{8} + \frac{\lambda}{e^{2\pi}-1}\right)b = 0 \end{cases} \quad \text{vilket ger } \lambda = \pm(e^{2\pi}-1)\frac{5}{32}.$$
   Detta ger $\|A\| = \frac{5}{32}(e^{2\pi}-1)$.
   Svar:
$$\|A\|_{C[0,\pi]\to C[0,\pi]} = \frac{1}{2}(e^{\frac{3\pi}{2}} + e^{\frac{\pi}{2}})$$
$$\|A\|_{L^2[0,\pi]\to L^2[0,\pi]} = \frac{5}{2}(2^{2\pi}-1)$$
   Dessutom är A kompakt operator betraktad som operator $C[0,\pi] \to C[0,\pi]$ och $L^2[0,\pi] \to L^2[0,\pi]$. Detta följer av
$$|Au(x_1) - Au(x_2)|^2 \leq \int_0^\pi |a(x_1,y) - a(x_2,y)|^2 dy\|u\|^2_{L^2[0,\pi]},$$
   Arzela-Ascoli sats och inbäddningen $\|u\|_{L^2[0,\pi]} \leq \sqrt{\pi}\|u\|_{C[0,\pi]}$.

2. Beräkna greenfunktionen $g(x,y)$ till differentialoperatorn $Lu = u'' - u$ med randvillkoren $R_1u = u(0) = 0, R_2u - u(0) = 0. u_1(x) = e^x, u_2(x) = e^{-x}$ är en bas för $\mathcal{N}(L)$ och ansättningen $\theta(x,t) = a_1(t)u_1(x) + a_2(t)u_2(x)$, där $e(t,t) = 0, e'_x(t,t) = 1$, ger fundamentallösningen $e(x,t) = \sinh(x-t)$. Vi noterar att $g(x,t) = e(x,t)\theta(x-t)$
   satisfierar randvillkoren för $t \in (0,1)$. Alltså ges lösningen till $Lu = f, Ru = (R_1u, R_2u) = 0$ av
$$u(x) = \int_0^1 \sinh(x-t)\theta(x-t)f(t)dt = \int_0^x \sinh(x-t)f(t)dt.$$
   Definiera nu
$$T : C[0,1] \to C[0,1]$$

enligt $Tu(x) = -\int_0^x \sinh(x-t)\frac{1}{2}(1+(u(t^2)))dt$, som är en kontinuerlig funktion då integranden $\in C([0,1]\times[0,1])$. T är en kontraktion då

$$|Tu(x) - Tv(x)| = |\frac{1}{2}\int_0^x \sinh(x-t)((u(t^2)) - (v(t^2)))dt| \le$$
$$\le \frac{1}{2}\int_0^x \sinh(x-t)dt\|u-v\|_\infty = \frac{1}{2}(\cosh x - 1)\|u-v\|_\infty \le$$
$$\le \frac{(e-1)^2}{2e}\|u-v\|_\infty \text{ eftersom } \frac{(e-1)^2}{2e} < 1.$$

Banachs fixpunktssats ger existens av entydigt bestämd fixpunkt, vilken också är den entydigt bestämd lösningen till differentialekvationsproblemet.

3. Antag att det finns $S, T \in \mathcal{B}(E)$ sådana att $ST - TS = I$. Detta medför att $TST - T^2S = T = ST^2 - TST$ vilket ger $2T = ST^2 - T^2S$. P.s.s. följer $nT^{n-1} = ST^n - T^nS$ för alla positiva heltal n. Vidare fås då $\|AB\| \le \|A\|\|B\|$ för alla $A, B \in \mathcal{B}(E)$ att

$$n\|T^{n-1}\| \le \|S\|\,\|T\|\,\|T^{n-1}\| + \|T^{n-1}\|\,\|T\|\,\|S\| \quad n = 2, 3, \ldots$$

Då $\|S\|, \|T\| < \infty$ följer att $\|T^{n-1}\| = 0$ för n tillräckligt stort, dvs $T^{n_0-1} = \mathbf{0} \in \mathcal{B}(E)$ för något positivt heltal.

Men $nT^{n-1} = ST^m - T^nS$ tillämpad på $n = n_0, n_0 - 1, \ldots, 2$ ger $T = \mathbf{0}$. Detta motsäger att $ST - TS = I$. Alltså saknas $S, T \in \mathcal{B}(e)$ med egenskapen ovan.

4. & 5. Kursboken...

6. Antag $T \in \mathcal{B}(H)$ normal och $x \in H$. Då gäller

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^Tx, x\rangle = \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2$$

och $\|Tx\| = \|T^*x\|$ följer. a) visad.

Av a) följer att $\mathcal{N}(T) = \mathcal{N}(T^*)$ för alla normala operator $T \in \mathcal{B}(H)$. Fixera att $\lambda \in \mathbb{C}$. Då är $\lambda I - T$ normal om T är normal ty $(\lambda I - T)^* = \bar{\lambda}I - T^*$ och

$$(\lambda I - T)(\bar{\lambda}I - T^*) = |\lambda|^2 I - \lambda T^* - \bar{\lambda}T + TT^* = \{TT^* = T^*T\} = (\bar{\lambda}I - T^*)(\lambda I - T).$$

Alltså gäller $\mathcal{N}(\lambda I - T) = \mathcal{N}(\bar{\lambda}I - T^*)$ dvs b) visad. Alternativt kan man bara “räkna på” från $\|T^*x - \bar{\lambda}x\|^2$.

Förslag till lösningar till tentamen i TMA400, 2000-05-30

Uppgift 1
Givet $Af(x) = \int_0^{2\pi} \cos(x-y) f(y)\, dy$, $0 \le x \le 2\pi$.
A är en begränsad linjär operator på $C[0,2\pi]$: Linjäriteten trivial (men bör visas). Begränsningen av A följer av

$$|Af(x)| = |\int_0^{2\pi} \cos(x-y) f(y)\, dy| \le \int_0^{2\pi} \underbrace{|\cos(x-y)|}_{\le 1} \underbrace{|f(y)|}_{\le \|f\|_\infty} dy \le$$
$$\le 2\pi \|f\|_\infty,$$

där $\|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|$. Alltså följer

$$\|Af\|_\infty \le 2\pi \|f\|_\infty,$$

vilket medför $\|A\| \le 2\pi$.
A är en begränsad linjär operator på $L^2[0,2\pi]$: Linjäriteten trivial som ovan. Begränsningen av A följer av

$$\int_0^{2\pi} |Af(x)|^2\, dx \le \int_0^{2\pi} (\int_0^{2\pi} |\cos(x-y)||f(y)|\, dy)^2\, dx \le$$
$$\le \{\text{Hölders olikhet}\} \le$$
$$\le \int_0^{2\pi} (\int_0^{2\pi} |\cos(x-y)|^2\, dy)(\int_0^{2\pi} |f(y)|^2\, dy)\, dx \le 4\pi^2 \|f\|_{L^2}^2,$$

där $\|f\|_{L^2} = (\int_0^{2\pi} |f(y)|^2\, dy)^{1/2}$. Alltså följer

$$\|Af\|_{L^2} \le 2\pi \|f\|_{L^2},$$

vilket medför $\|A\| \le 2\pi$.
$\|A\|_{C[0,2\pi]\to C[0,2\pi]} = 4$: Vi noterar att

$$\|A\|_{C[0,2\pi]\to C[0,2\pi]} \le \int_0^{2\pi} |\cos(x-y)|\, dy = \int_0^{2\pi} |\cos(y)|\, dy = 4.$$

För $n = 1, 2, \ldots$, låt f_n vara kontinuerliga funktioner på $[0,2\pi]$ som uppfyller $\|f_n\|_\infty = 1$ och dessutom $= 1$ på intervallen $[0, \frac{\pi}{2} - \frac{1}{n}] \bigcup [\frac{3\pi}{2} + \frac{1}{n}, 2\pi]$ och $= -1$ på intervallet $[\frac{\pi}{2} + \frac{1}{n}, \frac{3\pi}{2} - \frac{1}{n}]$ (här kan f_n t.ex. väljas som styckvis linjära funktioner). Då gäller

$$Af_n(0) \le 4$$

samt

$$Af_n(0) \ge \int_0^{2\pi} |\cos(y)|\, dy - 2 \cdot \frac{2}{n} = 4 - \frac{4}{n},$$

vilket visar påståendet ovan.
$\|A\|_{L^2[0,2\pi]\to L^2[0,2\pi]} = \pi$: Vi noterar att A är en kompakt självadjungerad operator på Hilbertrummet L^2, varför $\|A\|_{L^2[0,2\pi]\to L^2[0,2\pi]} = \sup\{|\lambda| : \lambda \text{ egenvärde till } A\}$. Eftersom $Af(x) = a\cos x + b \sin x$, där $a, b \in \mathbb{R}$ beror på f, ges varje egenfunktion på denna form. Liten kalkyl ger att λ är ett egenvärde till A om ekvationssystemet

$$A(a\cos(\cdot) + b\sin(\cdot))(x) = \lambda(a\cos(x) + b\sin(x))$$

i a, b har en icke-trivial lösning, dvs om $\lambda = \pi$.
Uppgift 2
Vi har $f \in C[0,1], \lambda \in \mathbb{R}$, där $|\lambda| < e(e-1)$, och ska visa att

$$\begin{cases} u''(x) + u'(x) + \lambda|u(x)| = f(x),\ x \in (0,1) \\ u(0) = u(1) = 0 \\ u \in C^2 \end{cases}$$

har en entydigt bestämd lösning.

1. Greenfunktioen till $\begin{cases} u'' + u' = F & x \in (0,1) \\ u(0) = u(t) = 0 \end{cases}$:

   Antag $e(x,t) = a_1(t) + a_2(t)e^{-x}$ uppfyller $\begin{cases} e(t,t) = 0 \\ e_x'(t,t) = 1 \end{cases}$. Detta ger $e(x,t) = 1 - e^{t-x}$. Greenfunktionen ges av

$$g(x,t) = \theta(x-t)(1-e^{t-x}) + b_1(t) + b_2(t)e^{-x}.$$

   Villkoren $g(0,t) = g(1,t)$, $0 < t < 1$ ger

$$\begin{cases} b_1(t) + b_2(t) = 0 \\ 1 - e^{t-1} + b_1(t) + b_2(t)e^{-1} = 0,\ 0 < t < 1. \end{cases}$$

   Alltså $g(x,t) = \theta(x-t)(1-e^{t-x}) + \frac{e^t - e}{e-1} + \frac{e-e^t}{e-1}e^{-x}$. Vi noterar att $g(x,t) \le 0$ alla x, t.

2. Sätt

$$\begin{cases} (Tu)(x) = \displaystyle\int_0^1 g(x,t)(f(t) - \lambda|u(t)|)dt,\ 0 \le x \le 1 \\ u \in C[0,1] \end{cases}$$

   Det ursprungliga problemet har en unik lösning omm T har en unik fixpunkt.

   För $u, v \in C[0,1]$ gäller

$$|(Tu)(x) - (Tv)(x)| = |\int_0^1 g(x,t)(\lambda|v(t)| - \lambda|u(t)|)dt| \le$$
$$\le |\lambda| \int_0^1 |g(x,t)||u(t)| - |v(t)||dt \le |\lambda| \int_0^1 |g(x,t)|dt\|u-v\|_\infty$$

   Sätt $j(x) = \int_0^1 |g(x,t)|dt$. Här är $j(x) = \int_0^1 g(x,t)(-1)dt$, lösningen till $j'' + j' = -1$ med randvillkoren $j(0) = j(1) = 0$. Alltså $j(x) = \frac{e}{e-1} - x - \frac{e}{e-1}e^{-x}$ med $j_{\max} = j(\ln \frac{e}{e-1}) = \frac{1}{e-1} + \ln(1 - \frac{1}{e}) \le \frac{1}{e-1} - \frac{1}{e} = \frac{1}{e(e-1)}$.

   Härav följer $\|Tu - Tv\|_\infty \le \frac{|\lambda|}{e(e-1)}\|u - v\|_\infty$, och Banach fixpunktssats medför att T har unik fixpunkt om $|\lambda| < e(e-1)$.

Uppgift 3
Givet avbildningen

$$T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, \frac{1}{2}(x_1 + x_2), \ldots, \frac{1}{n}(x_1 + \ldots x_n), \ldots).$$

Visa att

1. $T : \ell^2 \to \ell^2$ är en begränsad linjär operator
2. T ej är surjektiv

Linjäriteten hos T är trivial.
T begränsad operator: Tag $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$ och betrakta $\|T\mathbb{x}\|^2_{\ell^2}$. VLOG kan vi anta att $x_n \ge 0$ för alla n.

$$\|T\mathbb{x}\|^2_{\ell^2} = x_1^2 + \frac{1}{2^2}(x_1 + x_2)^2 + \ldots + \frac{1}{n^2}(x_1 + \ldots x_n)^2 + \ldots =$$
$$= \Sigma_{k=1}^\infty x_k^2 (\frac{1}{k^2} + \frac{1}{(k+1)^2} + \ldots) +$$
$$+ \Sigma_{k=1}^\infty \Sigma_{j=k+1}^\infty 2x_k x_j (\frac{1}{j^2} + \frac{1}{(j+1)^2} + \ldots).$$

Vidare gäller

$$\Sigma_{k=n}^\infty \frac{1}{k^2} \le \begin{cases} \int_{n-1}^\infty \frac{1}{x^2}\,dx = \frac{1}{n-1}, & n \ge 2 \\ 2 & n = 1 \end{cases}$$

vilket ger

$$\|T\mathbb{x}\|_{\ell^2}^2 \leq 2\|\mathbb{x}\|_{\ell^2}^2 + 2\Sigma_{k=1}^{\infty}\Sigma_{j=k+1}^{\infty} x_k x_j \frac{1}{j-1} \leq$$
$$\leq 2\|\mathbb{x}\|_{\ell^2}^2 + 4\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{x_k x_j}{k+j}.$$

Dessutom har vi

$$\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{x_k x_j}{k+j} = \Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\{(\frac{1}{k+j})^{1/2}(\frac{k}{j})^{1/4}x_k\}\{(\frac{1}{k+j})^{1/2}(\frac{j}{k})^{1/4}x_j\} \leq$$
$$\leq \{\text{Hölders olikhet}\} \leq$$
$$(\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{1}{k+j}(\frac{k}{j})^{1/2}x_k^2)^{1/2}(\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{1}{k+j}(\frac{j}{k})^{1/2}x_j^2)^{1/2}.$$

Här är

$$\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{1}{k+j}(\frac{k}{j})^{1/2}x_k^2 = \Sigma_{k=1}^{\infty}x_k^2\,\Sigma_{j=1}^{\infty}\frac{1}{k+j}(\frac{k}{j})^{1/2},$$

där

$$\Sigma_{j=1}^{\infty}\frac{1}{k+j}(\frac{k}{j})^{1/2} \leq \int_0^{\infty}\frac{1}{k+x}(\frac{k}{x})^{1/2}\,dx = \{y = \frac{x}{k}\} =$$
$$= \int_0^{\infty}\frac{1}{1+y}\frac{1}{\sqrt{y}}\,dy = C,$$

med C oberoende av k. Följdaktligen gäller

$$\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{\infty}\frac{x_k x_j}{k+j} \leq C\|\mathbb{x}\|_{\ell^2}^2$$

vilket medför att $\|T\mathbb{x}\|_{\ell^2} \leq \sqrt{2+4C}\|\mathbb{x}\|_{\ell^2}$.
T ej surjektiv: Vi noterar att $T : \ell^2 \to \ell^2$ är injektiv, dvs $T\mathbb{x}_1 = T\mathbb{x}_2 \Rightarrow \mathbb{x}_1 = \mathbb{x}_2$, och begränsad. Om T är surjektiv så ger den inversa avbildningssatsen att T^{-1} är en begränsad linjär avbildning. Sätt

$$\mathbb{y}_n = (0, 0, \ldots, 0, \underbrace{1}_{\text{plats } n}, 0, \ldots).$$

Då gäller $\|\mathbb{y}_n\|_{\ell^2} = 1$ och

$$T^{-1}\mathbb{y}_n = (0, 0, \ldots, 0, \underbrace{n}_{\text{plats } n}, \ldots),$$

dvs $\|T\mathbb{y}_n\|_{\ell^2} \geq n$, för alla n. Detta medför $\|T^{-1}\| = \infty$, vilket ger en motsägelse. Alltså T är ej surjektiv. (Alternativt kan man notera att följden

$$\mathbb{y} = (1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \ldots) \in \ell^2$$

medan den enda sekvens $\mathbb{x}$ sådan att $T\mathbb{x} = \mathbb{y}$ ges av

$$\mathbb{x} = (1, -1, 1, -1, 1, -1, \ldots) \notin \ell^2.)$$

Uppgift 4 & 5
Se kursboken.
Uppgift 6
Låt T vara en begränsad linjär operator på ett Hilbertrum H med $\|T\| = 1$. Antag att $Tx_0 = x_0$ för ett $x_0 \in H$. Ska visa att $T^*x_0 = x_0$.
Betrakta

$$\|T^*x_0 - x_0\|^2 = \langle T^*x_0 - x_0, T^*x_0 - x_0\rangle =$$
$$= \|T^*x_0\|^2 - \langle T^*x_0, x_0\rangle - \langle x_0, T^*x_0\rangle + \|x_0\|^2 =$$
$$= \|T^*x_0\|^2 - \langle x_0, Tx_0\rangle - \langle Tx_0, x_0\rangle + \|x_0\|^2 =$$
$$= \|T^*x_0\|^2 - \|x_0\|^2 \leq (\|T^*\|^2 - 1)\|x_0\|^2 =$$
$$= (\|T\|^2 - 1)\|x_0\|^2 = 0.$$

Alltså gäller $T^*x_0 = x_0$.