CHALMERS | GÖTEBORGS UNIVERSITET
Matematiska Institutionen
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# TMA401 Functional Analysis
# MAN670 Applied Functional Analysis
# 4th quarter 2006/2007

All document concerning the course can be found on the course home page:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma401/

# Home Assignment 1

**Problem 1:** Let $(E, \|\cdot\|)$ be a normed space and let $x, y$ be two vectors in $E$. Does

$$\|x+y\| = \|x\| + \|y\|$$

imply that one of the vectors is a scalar multiple of the other? Answer the question for $l^1$ and $l^2$.

**Problem 2:** Assume that $p \in (1, \infty)$ and let $q$ be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$\|\mathbb{x}\|_{l^p} = \sup\{\Sigma_{n=1}^{\infty}|x_n y_n| : \mathbb{y} \in l^q \text{ with } \|\mathbb{y}\|_{l^q} = 1\}.$$

**Problem 3:** Show that for every $g \in C([0,1])$ the mapping $T : C([0,1]) \to \mathbb{R}$ defined by

$$Tf = \int_0^1 f(t)g(t)\,dt$$

is linear and bounded. Moreover, calculate the operator norm $\|T\|$. (All functions are considered to be real-valued and $C([0,1])$ is equipped with the sup-norm)

**Problem 4:** Consider the vector space $l^1$ and set $\|\mathbb{x}\|_* = 2|\Sigma_{n=1}^{\infty} x_n| + \Sigma_{n=2}^{\infty}(1+\frac{1}{n})|x_n|$ for $\mathbb{x} = (x_1, x_2, \ldots, x_n, \ldots) \in l^1$. Show that $\|\mathbb{x}\|_*$ defines a norm on $l^1$ and that the vector space $l^1$ is a Banach space with this norm. Is this norm equivalent to the standard norm $\|\mathbb{x}\|_{l^1}$?

**Problem 5:** Let $F$ be a compact set in a normed space $(E, \|\cdot\|)$ and let $T : F \to F$ have the property

$$\|T(x) - T(y)\| < \|x - y\|, \quad \text{all } x, y \in F \text{ where } x \neq y.$$

Show that $T$ has a unique fixed point.

**Problem 6:** Let $\mathcal{B}$ be a basis in an infinite-dimensional Banach space $(E, \|\cdot\|)$. Show that $\mathcal{B}$ cannot be countable.

The solutions should be handed in at the latest on **Tuesday May 8th, 2007**.