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Peter Kumlin
Mathematics
Chalmers & GU

1 A Note on Spectral Theory

1.1 Introduction

In this course we focus on equations of the form

$$Af = g, \tag{1}$$

where A is a linear mapping, e.g. an integral operator, and g is a given element in some normed space which "almost everywhere" in text is a Banach space. The type of questions one usually and naturally poses are:

1. Is there a solution to $Af = g$ and, if so is it unique?
2. If the RHS g is slightly perturbed, i.e. data $\tilde{g}$ is chosen close to g, will the solution $\tilde{f}$ to $A\tilde{f} = \tilde{g}$ be close to f?
3. If the operator $\tilde{A}$ is a good approximation of the operator A will the solution $\tilde{f}$ to $\tilde{A}\tilde{f} = g$ be a good approximation for f to $Af = g$?

These questions will be made more precise and to some extent answered below.

One direct way to proceed is to try to calculate the "inverse operator" A^{-1} and obtain f from the expression

$$f = A^{-1}g.$$

Earlier we have seen examples of this for the case where A is a small perturbation of the identity mapping on a Banach space X, more precisely for $A = I + B$ with $\|B\| < 1$. Here the inverse operator A^{-1} is given by the Neumann series

$$\Sigma_{n=0}^{\infty}(-1)^n B^n,$$

(B^0 should be interpreted as I). We showed that if $B \in \mathcal{B}(X, X)$, where X is a Banach space, then we got $C \equiv \Sigma_{n=0}^{\infty}(-1)^n B^n \in \mathcal{B}(X, X)$ and

- $(I + B)C = C(I + B) = I_X$
- $\|C\| \leq \frac{1}{1-\|B\|}$

Sometimes we are able to prove the existence and uniqueness for solutions to equations $Af = g$ in Banach spaces X without being able to explicitly calculate the solution. An example is as follows: *Assume there exists a family* $\{A_t\}_{t\in[0,1]}$ *of bounded linear operators on a Banach space that satisfies the following conditions: There exists a positive constant* C *such that*

1. $\|f\| \leq C\|A_t f\|$ *for all* $t \in [0,1]$ *and all* $f \in X$

2. $\|A_t f - A_s f\| \leq C|t-s|\,\|f\|$ *for all* $t, s \in [0,1]$ *and all* $f \in X$

3. A_0 *is an invertible operator on* X*, where the inverse is a bounded linear operator on* X*.*

Then we can conclude that also A_1 *is an invertible operator on* X*, where the inverse is a bounded linear operator on* X*.* This is a general method and is referred to as the **method of continuity**. The idea is here that A_1 is a perturbation of the "nice" invertible operator A_0, where the perturbation is controlled by conditions 1 and 2 above. The proof of the statement, essentially that A_1 is surjective, is given below and is based on Banach's fixed point theorem.

Before we proceed let us make a clarification concerning the use of different notions.

When we talk about an operator A we do not a priori assume that A is linear (even if it is a mapping between two vector spaces) though it is in most applications.

What do we mean by an "inverse operator"? If we consider A as a mapping $X \to Y$ it is enough for A to be injective, i.e. $A(x) = A(y)$ implies $x = y$, for A^{-1} to be defined as a mapping $\mathcal{R}(A) \to X$. Here $\mathcal{R}(A) = \{y \in Y : y = A(x) \text{ for some } x \in X\}$ is a subset of Y. The injectivity implies that the equation (1) has at most one solution, viz. if $g \in \mathcal{R}(A)$ there exists a unique solution otherwise there is no solution. Moreover if A is surjective, i.e. $\mathcal{R}(A) = Y$, then the equation has a unique solution f for every $g \in Y$. So if we consider A in the category of mappings we say that A^{-1}, called the **inverse mapping** to A, exists if the equation (1) has a unique solution for every RHS, i.e. $A^{-1}(f) = g$.

However if X and Y are normed spaces and A is a bounded linear mapping we could look for a mapping B such that

$$AB = I_{\mathcal{R}(A)}, \quad BA = I_X$$

with the additional properties to be linear (which actually is automatic, check it!) and bounded. Hence if we view A in the category of bounded linear operators we call a bounded linear mapping B satisfying the conditions above the **inverse operator** to A. Also in this case we could have that A is surjective, i.e. $\mathcal{R}(A) = Y$. In particular this is natural to assume in the case $X = Y$ if we view the operator A as an element in $\mathcal{B}(X, X)$ where X is a Banach space. We observe that the space $\mathcal{B}(X, X)$, for short denoted by $\mathcal{B}(X)$, is not just a Banach space but also a Banach algebra, i.e. there is a multiplication defined in $\mathcal{B}(X)$ given by composition of operators

$$ST(x) = S(Tx)$$

which satisfies the norm inequality

$$\|ST\| \leq \|S\|\|T\|.$$

The inverse operator for A, provided A is surjective, is the inverse element to A in the Banach algebra $\mathcal{B}(X)$.

In connection with the Neumann series technique let us consider the following example. Set

$$X = Y = \mathcal{P}([0,1])$$

and

$$Ap(x) = (1 - \frac{x}{2})p(x), \quad x \in [0,1].$$

Moreover assume that X and Y are equipped with the L^2–norm. This means that the normed spaces $(X, \|\cdot\|_2)$ and $(Y, \|\cdot\|_2)$ are not Banach spaces. If we complete the normed space we obtain the Banach space $L^2([0,1])$. The question is whether A is invertible or not? First we note that A is injective, i.e. $Ap = Aq$ implies $p = q$. This is straight-forward since $Ap = Aq$ in Y means that $Ap(x) = Aq(x)$ for all $x \in [0,1]$ and hence $p(x) = q(x)$ for all $x \in [0,1]$, i.e. $p = q$ in X. But A is not surjective since $\mathcal{R}(A)$ consists of all restrictions of polynomials with a zero at $x = 2$ to the interval $[0,1]$. This shows that $A : X \to \mathcal{R}(A)$ has an inverse mapping. Moreover we note that A is a bounded linear mapping from x into Y with the operator norm given by

$$\|A\| = \sup_{\substack{p \in \mathcal{P}([0,1]) \\ \|p\|_2 = 1}} (\int_0^1 |(1 - \frac{x}{2})\, p(x)|^2 \, dx)^{\frac{1}{2}} = 1.$$

Prove this! A question is now if A has an inverse operator. Since A is given as a multiplication mapping it is clear that the inverse mapping also is given by a multiplication mapping where the multiplier is $\frac{2}{2-x}$. We obtain $A^{-1} : \mathcal{R}(A) \to \mathcal{P}([0,1])$ as a bounded linear mapping with the operator norm

$$\|A^{-1}\| = \sup_{\substack{p \in \mathcal{R}(A) \\ \|p\|_2 = 1}} (\int_0^1 |(\frac{2}{2-x})\, p(x)|^2 \, dx)^{\frac{1}{2}} = 2.$$

Prove also this!

If we extend A to all of $L^2([0,1])$, call this extension $\tilde{A}$, which can be done uniquely since the polynomials in $\mathcal{P}([0,1])$ are dense in $L^2([0,1])$ and A is a bounded linear operator on $\mathcal{P}([0,1])$, we observe that $\|I - \tilde{A}\| < 1$, where $\|\cdot\|$ denote the operator norm on $L^2([0,1])$, since

$$\int_0^1 |\frac{x}{2} f(x)|^2 \, dx \le \frac{1}{4} \int_0^1 |f(x)|^2 \, dx,$$

and hence

$$\|(I - \tilde{A})f\| \le \frac{1}{2}\|f\|.$$

From this we get that the Neumann series $\Sigma_{n=0}^{\infty}(I - \tilde{A})^n$ gives an expression for the inverse mapping to $\tilde{A}$, since $\tilde{A}$ can be written as $\tilde{A} = I - (I - \tilde{A})$ on $L^2([0,1])$. It is no surprise that

$$\tilde{A}^{-1}p(x) = \Sigma_{n=0}^{\infty}(I - \tilde{A})^n p(x) = \Sigma_{n=0}^{\infty}(\frac{x}{2})^n p(x) = \frac{2}{2-x} p(x).$$

Observe that $(\tilde{A}|_{\mathcal{P}([0,1])})^{-1} = A^{-1}$. The Neumann series applied to an element of $\mathcal{R}(A)$ yields a polynomial, but to make sure that the series converges we need to consider the series in a Banach space and not just a normed space. Moreover we see that $\tilde{A}^{-1}$ is a bounded operator on the whole of $L^2([0,1])$ with the norm $\|\tilde{A}^{-1}\| = 2$ but A^{-1} is not a bounded operator on the whole of $\mathcal{P}([0,1])$ and this despite the fact

$$\|I - A\|_{X \to Y} \leq \|I - \tilde{A}\|_{L^2 \to L^2} < 1.$$

Below we present some observations that are related to the concepts inverse mapping/inverse operator.

- We first consider mappings on vector spaces. The following holds true.

  Theorem 1.1. *Assume that E is a finite-dimensional vector space and that $A : E \to E$ is a linear mapping. Then the following statements are equivalent:*

  1. *A is bijective*
  2. *A is injective, i.e. $\mathcal{N}(A) = \{0\}$*
  3. *A is surjective, i.e. $\mathcal{R}(A) = E$*

  Note that this is not true for infinite-dimensional vector spaces, which is shown by the following example.
  Set

  $$E = C([0,1])$$

  and

  $$Af(x) = \int_0^x f(t)\,dt, \quad x \in [0,1].$$

  Prove that A is injective but not surjective!

- *From now on we only consider linear mappings $X \to Y$ where X and Y are Banach spaces.* We know that

  - *A is continuous at $x_0 \in X$ implies that A is continuous on X*
  - *A is continuous iff A is a bounded mapping.*

  It can be shown, not without some effort, that there are linear mappings $A : X \to X$ that are not bounded, i.e. the linearity and the mapping property $A(X) \subset X$ is not enough for A to be a bounded operator. This has some relevance when returning to the **stability**-question 2 in the introduction, i.e. whether the fact that $A : X \to Y$ is a bijective bounded linear operator implies that there exists a constant C such that

  $$\|f - \tilde{f}\| \leq C\|g - \tilde{g}\|,$$

  for all $g, \tilde{g} \in Y$ where $Af = g$ and $A\tilde{f} = \tilde{g}$? The answer is given by

  Theorem 1.2 (Inverse mapping theorem). *Assume that $A : X \to Y$ is a bijective bounded linear mapping from the Banach space X onto the Banach space Y. Then the mapping A^{-1} exists as a bounded linear mapping from Y onto X.*

The answer to the question above is yes!

The proof is based on Baire's Theorem (see [4] section 1.4). Often the inverse mapping theorem is given as a corollary to the open mapping theorem, that also can be proved using Baire's theorem. We formulate the theorem without proof.

Theorem 1.3 (Open mapping theorem). *Assume that $A : X \to Y$ is a surjective bounded linear mapping from the Banach space X onto the Banach space Y. Then A maps open sets in X onto open sets in Y.*

Recall that a mapping $A : X \to Y$ is continuous iff the set $A^{-1}(U)$ is open in X for every open set U in Y.

It has been shown, using Neumann series, that the equation $Af = g$ is uniquely solvable if $A = I - T$ and $\|T\| < 1$. However this is a serious restriction. We want to solve equations where T is not a small perturbation of the identity mapping. To do this we will, as for the finite-dimensional case, study the equation

$$(\lambda I - T)f = g$$

where λ is a complex parameter. In this context concepts like spectrum, resolvent and resolvent set are introduced. A more extensive treatment can be found in the books [2], [3] and [5]. The first two books are on the same level as the textbook.

Assume that X is a complex normed space and that $T : \mathcal{D}(T) \to X$ is a bounded linear mapping with $\mathcal{D}(T) \subseteq X$. Often we have $\mathcal{D}(T) = X$.

Definition 1.1. *The* **resolvent set** *for T, denoted $\rho(T)$, consists of all complex numbers $\lambda \in \mathbb{C}$ for which $(T - \lambda I)^{-1}$ exists as an inverse operator on all of X. The mapping $\rho(T) \ni \lambda \mapsto (\lambda I - T)^{-1}$ is called the* **resolvent** *for T.*

It follows from the definition that $\lambda \in \rho(T)$ implies that $\mathcal{N}(T - \lambda I) = \{0\}$ and that $\mathcal{R}(T - \lambda I) = X$.

Definition 1.2. *The* **spectrum** *for T, denoted by $\sigma(T)$, is the set $\mathbb{C} \backslash \rho(T)$. This set is the union of the three mutually disjoint subsets $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$. These are called the* **point spectrum**, *the* **continuous spectrum** *and the* **residual spectrum** *respectively and are defined by the properties*

- $\lambda \in \sigma_p(T)$ *if $\mathcal{N}(T - \lambda I) \neq \{0\}$. Here λ is called an* **eigenvalue** *for T and a $v \in \mathcal{N}(T - \lambda I) \setminus \{0\}$ is called an* **eigenvector** *corresponding to the eigenvalue λ;*

- $\lambda \in \sigma_c(T)$ *if $\mathcal{N}(T - \lambda I) = \{0\}$ and $\mathcal{R}(T - \lambda I)$ is dense in X but $(T - \lambda I)^{-1}$ is not a bounded operator;*

- $\lambda \in \sigma_r(T)$ *if $\mathcal{N}(T - \lambda I) = \{0\}$ but $\mathcal{R}(T - \lambda I)$ is not dense in X.*

Examples:

1. Assume that $T : X \to X$ is a linear mapping on a finite-dimensional normed space X. Then we have $\sigma(T) = \sigma_p(T)$ and the spectrum consists of finitely many elements.

2. Consider the linear mapping $T : l^2 \to l^2$ defined by

$$(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, \ldots)$$

T is a so called right shift operator. Then we have $0 \in \sigma(T) \setminus \sigma_p(T)$.

From now on we assume that T is a bounded operator.

Theorem 1.4. *The resolvent set is an open set.*

Proof. (a sketch) We note that

- if $A : X \to X$ is a bounded linear operator with $\|A\| < 1$ then $(I - A)^{-1}$ exists as an inverse operator on all of X and

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

(Neumann series)

- if $\lambda_0 \in \rho(T)$ we have the formula

$$T - \lambda I = (T - \lambda_0)(I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}).$$

Combining these observations we obtain the result. □

In this context we give a proof for the method of continuity. Condition 1 implies that all A_t, $t \in [0, 1]$, are injective. Assuming that A_t has an inverse operator defined on all of X we can write the operator A_s as

$$A_s = A_t(I + A_t^{-1}(A_s - A_t)).$$

Hence it follows that A_s is invertible if $\|A_t^{-1}(A_s - A_t)\| < 1$. But now condition 1 implies that $\|A_t^{-1}\| \leq C$ and condition 2 implies $\|A_s - A_t\| \leq C|s - t|$. This yields that

$$\|A_t^{-1}(A_s - A_t)\| \leq \|A_t^{-1}\| \, \|A_s - A_t\| < 1$$

provided

$$|s - t| < \frac{1}{C^2}.$$

Take a finite sequence of points t_n, $0 = t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_N = 1$, such that

$$\max_{n=1,2,\ldots,N-1} |t_{n+1} - t_n| < \frac{1}{C^2}.$$

The argument above shows that $A_{t_{n+1}}$ is invertible if A_{t_n} is invertible and hence the invertibility of A_0 implies the invertibility of A_1. (Invertibility of an operator B means that B^{-1} exists as an inverse operator and B is surjective.)

Theorem 1.5. *The spectrum $\sigma(T)$ belongs to the disc*

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$$

in the complex plane.

Proof. Exercise! □

Theorem 1.6. *The spectrum $\sigma(T)$ is non-empty.*

The proof can be based on Liouville's Theorem, well-known from courses in complex analysis, but is omitted.

Definition 1.3. *The* **approximate point spectrum** *to T, denoted by $\sigma_a(T)$, consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n)_{n=1}^{\infty}$ in X, with $\|x_n\| = 1$ such that*

$$\lim_{n \to \infty} \|(T - \lambda I)x_n\| = 0.$$

The following result summarizes the important properties for the approximate point spectrum.

Theorem 1.7. *Assume that T is a bounded operator on X. Then we have:*

1. *$\sigma_a(T)$ is a closed non-empty subset of $\sigma(T)$;*
2. *$\sigma_p(T) \bigcup \sigma_c(T) \subset \sigma_a(T)$;*
3. *the boundary of $\sigma(T)$ is a subset of $\sigma_a(T)$;*

From now on we assume that the linear operator T is compact and that X is a Banach space. An operator T is called compact on X if for every bounded sequence $(x_n)_{n=1}^{\infty}$ in X there exists a convergent subsequence of $(Tx_n)_{n=1}^{\infty}$ in X. Using Riesz' Lemma (see [4] section 1.2) together with a lot of hard work one can show the following theorem that usually is called **Fredholm's alternative**.

Theorem 1.8 (Fredholm's alternative). *Let T be a compact linear operator on a Banach space X and let $\lambda \in \mathbb{C} \setminus \{0\}$. Then one of the statements below hold true:*

1. *the homogeneous equation*

   $$Tx - \lambda x = 0$$

   has non-trivial solutions $x \in X$

2. *for every $y \in X$ the equation*

   $$Tx - \lambda x = y$$

   has a unique solution $x \in X$.

In the second case the operator $(T - \lambda I)^{-1}$ exists as a bounded operator.

Example: Consider the Volterra equation

$$f(x) = g(x) + \int_0^x K(x,y) f(y)\, dy \quad 0 \le x \le 1,$$

where K is a continuous function for $0 \le x, y \le 1$. Show that for every $g \in C([0,1])$ there exists a $f \in C([0,1])$ that solves the equation. From Fredholm's alternative with $X = C([0,1])$ it is enough to show that $\mathcal{N}(T - I) = \{0\}$, where T is the compact (show this using for instance Arzela-Ascoli Theorem) operator

$$Tf(x) = \int_0^x K(x,y) f(y)\, dy$$

on $C([0,1])$. We will show that

$$f(x) = \int_0^x K(x,y) f(y)\, dy \quad 0 \le x \le 1$$

implies that $f = 0$. Set $M = \max_{0 \le x,y \le 1} |K(x,y)|$ and

$$\phi(x) = \int_0^x |f(y)|\, dy \quad 0 \le x \le 1.$$

It follows that ϕ is differentiable and

$$\phi^{'}(x) = |f(x)| \le M\phi(x) \quad 0 \le x \le 1$$

and hence $(\phi(x) e^{-Mx})' \le 0$ and finally

$$0 \le \phi(x) \le \phi(0) e^{-Mx} \quad 0 \le x \le 1.$$

But we have $\phi(0) = 0$ and the desired conclusion follows.

Moreover the following result holds.

Theorem 1.9 (Riesz-Schauder Theorem). *Assume that $T : X \to X$ is a compact linear operator on a Banach space X. Then the following statements hold true:*

1. *$\sigma_p(T)$ is countable, can be finite or even empty;*
2. *$\lambda = 0$ is the only clustering point for the set $\sigma_p(T)$;*
3. *λ is an eigenvalue if $\lambda \in \sigma(T) \setminus \{0\}$;*
4. *X infinite-dimensional space implies that $0 \in \sigma(T)$;*
5. *For $\lambda \ne 0$ the subspaces $\mathcal{R}((T - \lambda I)^r)$ are closed and the subspaces $\mathcal{N}((T - \lambda I)^r)$ are finite-dimensional for $r = 1, 2, 3, \ldots$;*

6. *For $\lambda \neq 0$ there exists a non-negative integer r, depending on λ, such that*

$$X = \mathcal{N}((T-\lambda)^r) \bigoplus \mathcal{R}((T-\lambda)^r)$$

and

$$\mathcal{N}((T-\lambda I)^r) = \mathcal{N}((T-\lambda I)^{r+1}) = \mathcal{N}((T-\lambda I)^{r+2}) = \cdots$$

and

$$\mathcal{R}((T-\lambda I)^r) = \mathcal{R}((T-\lambda I)^{r+1}) = \mathcal{R}((T-\lambda I)^{r+2}) = \cdots .$$

Moreover if $r > 0$ it holds that

$$\mathcal{N}(I) \subset \mathcal{N}((T-\lambda I)^1) \subset \cdots \subset \mathcal{N}((T-\lambda I)^r)$$

and

$$\mathcal{R}(I) \supset \mathcal{R}((T-\lambda I)^1) \supset \cdots \supset \mathcal{R}((T-\lambda I)^r),$$

where $\subset$ and $\supset$ here denotes proper subset.

7. *For $\lambda \neq 0$ it holds that*[1]

$$\mathcal{R}(T-\lambda I) = \mathcal{N}(T^* - \lambda I)^{\perp}.$$

The last statement in the theorem has a meaning to us if X is a Hilbert space (the "Riesz part" of the theorem) but it is also possible to assign a meaning to the concept adjoint operator in a Banach space and to the "orthogonal complement" that usually is called the set of annihilators (the "Schauder part" of the theorem is the generalisation to arbitrary Banach spaces). It should be noted that the definition of adjoint operator on a Banach space differs slightly from the Hilbert space case but just up to an isometry. For those who are interested we refer to [2], [3] and [5].

If we use the last part of Riesz-Schauder's Theorem we can make Fredholm's alternative a bit more precise.

Theorem 1.10 (Fredholm's alternative). *Let T be a compact linear operator on a Banach space X and let $\lambda \neq 0$. Then it holds that $Tx - \lambda x = y$ has a solution iff*[2] *$y \in \mathcal{N}(T^* - \lambda I)^{\perp}$.*

Now let $X = H$ be a Hilbert space and T a compact linear operator on H. If T is self-adjoint we obtain the counterpart to Fredholm's alternative that is given in the textbook [1] theorem 5.2.6, which using Hilbert space notations can be written as

$$\mathcal{R}(T-I) = \mathcal{N}(T-I)^{\perp}.$$

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[1]There is a difference here compared to when X is a Hilbert space which depends on the definition of adjoint operator. If we use our standard definition from [1] the relation should be

$$\mathcal{R}(T-\lambda I) = \mathcal{N}(T^* - \overline{\lambda} I)^{\perp}.$$

[2]If X is a Hilbert space and the usual definition for adjoint operator is used the relation should be $y \in \mathcal{N}(T^* - \overline{\lambda} I)^{\perp}$.

For the case with self-adjoint compact operators on Hilbert spaces the integer r in Theorem 1.9 will be equal to 1. In connection with $n \times n$–matrices and their eigenvalues this corresponds to the fact that the algebraic multiplicity and the geometric multiplicity are equal for eigenvalues to hermitian matrices.

Let us very briefly indicate the Banach space case.

For arbitrary Banach spaces X we set $X^* = \mathcal{B}(X, \mathbb{C})$, considered as a Banach space with the norm given by the operator norm $\|\cdot\|_{X \to \mathbb{C}}$. Let T be a bounded linear mapping from the Banach space X into the Banach space Y. We define the mapping $T^* : Y^* \to X^*$ using the relation

$$(T^* y^*)(x) = y^*(Tx) \quad \text{alla} \quad y \in Y^*,\ x \in X.$$

It is easy to show that T^* is a bounded linear mapping with $\|T^*\|_{Y^* \to X^*} = \|T\|_{Y \to X}$. For sets $A \subset X$ and $B \subset X^*$ in a Banach space X we set

$$A^{\perp} = \{x^* \in X^* : x^*(x) = 0 \quad \text{alla} \quad x \in A\}$$

and

$$B^{\perp} = \{x \in X : x^*(x) = 0 \quad \text{alla} \quad x^* \in A\}.$$

Here $A^{\perp}$ and $B^{\perp}$ become closed subspaces in X^* and X respectively. We detect a difference in the definition compared to the orthogonal complement for a set A in a Hilbert space! The following result can be proved (we recognise it for the case $X = \mathbb{C}^n, Y = \mathbb{C}^m$ and T given by a $m \times n$–matrix).

Theorem 1.11. *Assume that X and Y are Banach spaces and that $T \in \mathcal{B}(X, Y)$. Then it holds that*

$$\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^{\perp}.$$

If $\overline{\mathcal{R}(T)} = \mathcal{R}(T)$ it holds that

$$\overline{\mathcal{R}(T^*)} = \mathcal{N}(T)^{\perp}$$

and $\overline{\mathcal{R}(T^*)} = \mathcal{R}(T^*)$.

For T in Theorem 1.11 it is true that if T is compact then T^* is also compact (the converse is also true). T being compact also implies that $\mathcal{R}(T - \lambda I)$ is closed (compare [4] section 1.6). Theorem 1.11 implies that

$$\mathcal{R}(T - \lambda I) = \mathcal{N}(T^* - \lambda I)^{\perp}$$

and

$$\mathcal{R}(T^* - \lambda I) = \mathcal{N}(T - \lambda I)^{\perp}.$$

Finally we refer to the textbook [1] for the spectral theory for compact self-adjoint operators.

References

[1] L.Debnath/P.Mikusinski, *Introduction to Hilbert Spaces with Applications 3rd ed.*, Academic Press 2005

[2] A.Friedman, *Foundations of modern analysis*, Holt Rinehart and Winston, 1970

[3] E.Kreyszig, *Introduction to functional analysis with applications*, Wiley 1989

[4] P.Kumlin, *Exercises*, Mathematics, Chalmers & GU 2006/2007

[5] W.Rudin, *Functional Analysis*, McGraw-Hill, 1991