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1 A Note on L^p–spaces

1.1 Introduction

A basic feature for the important results in this course – Banach's fixed point theorem, Brouwer's fixed point theorem, Schauder's fixed point theorem, Hilbert-Schmidt theorem and others – is that the mappings appearing should be defined on complete normed spaces, i.e. on **Banach spaces**. The completeness is crucial and the theorems would no longer be true without the assumption on completeness.

A technique often used to prove the existence of a solution to a problem (and also to find the solution) is to find solutions to approximate problems and by improving the approximations it can sometimes be possible to obtain a sequence of approximative solutions that forms a Cauchy sequence in a proper space. A solution to the original problem can then often be obtained as the limit to the Cauchy sequence provided the space is a Banach space.

An example of a function space that has been treated is the vector space consisting of the continuous functions defined on $\mathbb{R}^n$ or some "nice"[1] subset Ω of $\mathbb{R}^n$, with pointwise defined addition and multiplication by scalars. We note the if $C(\Omega)$ is equipped with the sup-norm, i.e.

$$\|f\| = \sup_{t\in\Omega} |f(t)|, \quad f \in C(\Omega),$$

then the normed space $(C(\Omega), \|\cdot\|)$ becomes a Banach space. But if $C(\Omega)$ is supplied with the norm

$$\|f\|_1 = \int_\Omega |f(t)|\, dt, \quad f \in C(\Omega),$$

then $(C(\Omega), \|\cdot\|_1)$ is a normed space but **not** a Banach space. See for instance example 1 below. The set Ω is supposed to be a compact subset of $\mathbb{R}^n$ so all integrals are finite. It is a pity that $(C(\Omega), \|\cdot\|_1)$ is not a Banach space since the norm $\|\cdot\|_1$ is a natural measure of size. If f is a density function then $\|f\|_1$ corresponds to the total mass. Moreover in physics the integral

$$\|f\|_2 = (\int_\Omega |f(t)|^2\, dt)^{1/2}, \quad f \in C(\Omega).$$

measures the "energy" of a system described by f. In general it is natural to consider norms

$$\|f\|_p = (\int_\Omega |f(t)|^p\, dt)^{1/p}, \quad f \in C(\Omega),$$

[1] We assume that Ω is compact and equal to the closure of its interior.

where $p \in [1, \infty)$. To see that these expressions really define norm functions can be done with same technique that was used to prove the corresponding statements for the sequence spaces l^p.

Example 1: Consider the set $\Omega = [0, 1] \subset \mathbb{R}$ and define

$$f_n(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}) \\ 2n(t - \frac{1}{2}) & t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}) \\ 1 & t \in [\frac{1}{2} + \frac{1}{2n}, 1] \end{cases}$$

for $n = 1, 2, \ldots$ Sketch the graph for f_n here!

We see that $(f_n)_{n=1}^{\infty}$ defines a Cauchy sequence in the normed space $(C[0, 1], \| \ \|_1)$ since

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{1}{2\min(n,m)}} |f_n(t) - f_m(t)|\, dt \\ &\leq \frac{1}{2\min(n, m)} \to 0, \quad n, m \to \infty. \end{aligned}$$

However there is no continuous function f such that $f_n \to f$ in $(C[0, 1], \| \ \|_1)$. Prove this! On the other hand the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to h given by

$$h(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ 1 & t \in (\frac{1}{2}, 1] \end{cases} .$$

h is not continuous but still Riemann integrable and satisfies

$$\lim_{n \to \infty} \|f_n - h\|_1 = 0.$$

The fact that the function h above is Riemann integrable might suggest that the normed space

$$(\text{Riemann integrable functions}, \| \cdot \|_1) \tag{1}$$

is a Banach space. It is clear that linear combinations of Riemann integrable functions are Riemann integrable and that also products of Riemann integrable functions are Riemann integrable[2]. However Riemann integrable functions are not closed under pointwise limits as seen from the following example.

[2] If f is Riemann integrable then so is f^2 and if both f and g are Riemann integrable then so is fg since $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$.

Example 2: Let Ω denote the interval $[0,1]$ and let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of all rational numbers in the interval $[0,1]$. For $n = 1, 2, \ldots$ define

$$f_n(t) = \chi_{\{r_1,\ldots,r_n\}}(t) = \left\{ \begin{array}{ll} 1 & t \in \{r_1, \ldots, r_n\} \\ 0 & t \notin \{r_1, \ldots, r_n\} \end{array} \right. .$$

Moreover set

$$f(t) = \chi_{\{r_1,r_2,r_3,\ldots\}}(t) = \left\{ \begin{array}{ll} 1 & t \in \{r_1, r_2, r_3, \ldots\} \\ 0 & t \notin \{r_1, r_2, r_3, \ldots\} \end{array} \right. .$$

We note that f_n is Riemann integrable for every n and that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in the normed space (Riemann integrable functions, $\|\cdot\|_1$), but the pointwise limit function f is not Riemann integrable. Prove this! This implies that if we want to have a Banach space containing all Riemann integrable functions we ought accept f as an element in that space since it is the pointwise limit of the sequence $(f_n(x))$ (we also have $0 \leq f_n(x) \uparrow f(x) \leq 1$ for all $x \in [0,1]$). In applications it will be important for us to have strong convergence theorems of the form

$$\lim_{n\to\infty} \int f_n \, dx = \int \lim_{n\to\infty} f_n \, dx.$$

At the same time we note that $f \neq \mathbf{0}$, where $\mathbf{0}$ denotes the function that is pointwise 0 for all $x \in [0,1]$, but nether the less we have $\|f - \mathbf{0}\|_1 = 0$. This means that we can not detect the difference between f and $\mathbf{0}$ measuring with the $\|\cdot\|_1$–norm and have to **identify** these functions. We say that the functions differs on a set of measure 0. This identification also has to be done for Riemann integrable functions for $\|\cdot\|_1$ to be a norm.

Considering the $\|\cdot\|_p$–norms, $1 \leq p < \infty$ in general, it can be observed that only for $p = 2$ the norm is a Hilbert space norm, i.e. there can be defined an inner product $\langle\cdot,\cdot\rangle$ on the vector space such that the following relation between the inner product and the norm holds true: $\|x\| = \sqrt{\langle x, x\rangle}$ for all x. Neither the sup-norm can be connected with an inner product. The Hilbert space structure will be important to us in connection with spectral theory in chapter 4 in [2]. However $\|\cdot\|_2$ will be a Hilbert space norm.

The problem is now to extend the normed space $(C(\Omega), \|\cdot\|_p)$ in such a way that we obtain a Banach space. The method to complete the space that is discussed in section 4 chapter 1 in [2] has the disadvantage that the properties of the elements in the completion can be hard to read off and it is not obvious that the elements are pointwise defined functions.

We will below in very few words discuss the main properties of the Lebesgue integrals and the L^p–spaces. Some ideas for the proofs will be sketched. For those who are interested in a thorough treatment we refer to the books by Folland [3] (textbook on graduate level), Rudin [5] (also a graduate level textbook), Rudin [6] (a more elementary book), Apostol [1] (has been used for undergraduate courses at GU) or why not Hörmander [4]. The presentations differ slightly but most are based on measure theory.

1.2 Lebesgue measure on $\mathbb{R}^n$

In measure theory we want to generalize the concept of length of an interval in $\mathbb{R}$, area of a rectangle in $\mathbb{R}^2$ on so on to a wider class of sets. The ultimate goal is to assign a measure to as many sets as possible where the measure has to satisfy certain natural conditions. If all intervals $[a,b] \subset \mathbb{R}$, $a < b$ (as well as the intervals $[a,b)$, $(a,b]$, (a,b)) should have the measure $b-a$ then there are subsets of the real numbers that are impossible to assign a measure to[3]. This is hard to prove and is based on the axiom of choice[4].

First let us see for which subsets of a an arbitrary set X it would be natural to be able to assign measure to. Intuitively it is natural that given countable many sets, where all have a well-defined measure, all sets that can be obtained by countably many applications with the set operations union, intersection and complement should also be possible to assign a measure to. This motivates the following definition.

Definition 1.1. *A set $\mathcal{M}$ of subsets of X is called a σ–algebra if*

1. $\emptyset \in \mathcal{M}$
2. $E \in \mathcal{M}$ *implies* $X \setminus E \in \mathcal{M}$
3. $E_1, E_2, \ldots \in \mathcal{M}$ *implies* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$

A set in $\mathcal{M}$ is called a measurable set. Let $\mathcal{B}_n$ denote the smallest σ–algebra that contains all open sets in $\mathbb{R}^n$. This is called the **Borel σ–algebra**. For simplicity we restrict to the case $n = 1$ but what is said holds true for general n. There exists such a smallest σ–algebra, since the intersection of any collection of σ–algebras is a σ–algebra, and all the intervals of the four types above are contained here.

Given a σ–algebra $\mathcal{M}$ we can talk about a measure μ on $\mathcal{M}$. A measure should satisfy some properties encoded in the next definition.

Definition 1.2. *A measure μ on the σ–algebra $\mathcal{M}$ is a mapping*

$$\mu : \mathcal{M} \to [0, +\infty]$$

such that

1. $\mu(\emptyset) = 0$
2. $E_1, E_2, \ldots \in \mathcal{M}$ *mutually disjoint sets implies* $\mu(\bigcup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} \mu(E_n)$

[3] A well-known example of this is due to Vitali. Even more striking is the following example in $\mathbb{R}^3$ by Banach and Tarski and which only involves finite additivity. They proved: The unit ball in $\mathbb{R}^3$ can be decomposed into a finite number of pieces which may be reassembled, using only translation and rotation, to form 2 disjoint copies of the unit ball

[4] The axiom of choice says that for every class of non-empty sets E_λ, $\lambda \in \Lambda$, there exists a set consisting of one element from every set E_λ.

The property 2. is called countable additivity for measures and is a key property when defining Lebesgue integrals. The main question is now whether it is possible to prove the existence of a unique measure on the Borel σ–algebra with the property that all intervals with end points at a and b has the measure $|b-a|$. The answer is yes and this measure is called the **Borel measure**. This is the foundation on which the L^p-theory rests. The **Lebesgue measure** is obtained by completing the Borel measure in the following sense.

Definition 1.3. *Let μ be a measure on a σ–algebra $\mathcal{M}$. Then there exists a σ–algebra $\bar{\mathcal{M}}$ and a well-defined measure $\bar{\mu} : \bar{\mathcal{M}} \to [0, +\infty]$ such that $E \in \bar{\mathcal{M}}$ iff $E = A \bigcup B$, where $A \in \mathcal{M}$ and $B \subset C \in \mathcal{M}$ with $\mu(C) = 0$, and $\bar{\mu}(E) = \mu(A)$.*

What has been done is to add all subsets of measurable sets with measure 0 in such a way that also $\bar{\mathcal{M}}$ becomes a σ–algebra. Note that it follows from the definition that if $A, B \in \mathcal{M}$, $A \subset B$, then we have $\mu(A) \leq \mu(B)$. We call $\bar{\mathcal{B}}_1$ the **Lebesgue σ–algebra** on $\mathbb{R}$ and denote it by $\mathcal{L}_1$ and the completed Borel measure on $\mathcal{L}_1$ denoted m is called the **Lebesgue measure**.

We mentioned above that there are subsets of $\mathbb{R}$ that are not Lebesgue measurable. The following result can be proved.

Theorem 1.1 (Approximation). *Let $E \subset \mathbb{R}$ be Lebesgue measurable. Then we have*

$$m(E) = \inf\{m(U) : E \subset U,\ U \ \text{ open}\} = \sup\{m(K) : K \subset E,\ K \ \text{ compact}\}.$$

Moreover if $m(E) < \infty$ then for every $\epsilon > 0$ there exists an open set A consisting of finitely many open intervals such that

$$m((E \setminus A) \bigcup (A \setminus E)) < \epsilon.$$

What has been said about $\mathbb{R}$ is true for $\mathbb{R}^n$, $n = 2, 3, \ldots$, provided intervals are replaced by rectangles parallel to the axis etc. By $\mathcal{L}_n = \mathcal{L}$ we denote the Lebesgue σ–algebra on $\mathbb{R}^n$, i.e. the completed Borel σ–algebra $\mathcal{B}_n$, and the approximation theorem above corresponds to a natural generalization for $\mathbb{R}^n$.

1.3 Lebesgue measurable functions

We will now consider functions f that takes values in $\bar{\mathbb{R}} = \mathbb{R} \bigcup \{\pm\infty\}$ where we define $0 \cdot \infty = 0$. What has to be avoided is undefined expressions like $\infty - \infty$. In this section every function takes values in $\bar{\mathbb{R}}$. We say that the function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is **Lebesgue measurable** if $f^{-1}([a, \infty)) \in \mathcal{L}$ for every $a \in \mathbb{R}$. Here $f^{-1}(U)$ denotes the set $\{x \in \mathbb{R}^n : f(x) \in U\}$, i.e. the inverse image of U under f. From this definition it follows that $f^{-1}(E) \in \mathcal{L}$ for every Borel set E but also that all functions that can be formed using the operations

$$+ \quad \cdot \quad \sup_{n=1,2,\ldots} \quad \inf_{n=1,2,\ldots} \quad \limsup_{n=1,2,\ldots} \quad \liminf_{n=1,2,\ldots}$$

on Lebesgue measurable functions are Lebesgue measurable. More precisely, given Lebesgue measurable functions f, g, f_n, $n = 1, 2, \ldots$ then the functions

1. $f + g$, fg, λf, where $\lambda \in \mathbb{R}$
2. $\max(f, g)$, $\min(f, g)$
3. $\sup_{n=1,2,\ldots} f_n$, $\inf_{n=1,2,\ldots} f_n$
4. $\limsup_{n\to\infty} f_n \equiv \lim_{k\to\infty} \sup_{n\geq k} f_n$, $\liminf_{n\to\infty} f_n \equiv \lim_{k\to\infty} \inf_{n\geq k} f_n$

are also Lebesgue measurable. It can be shown that every continuous function is Lebesgue measurable! The most important examples of Lebesgue measurable functions are the so called **simple functions** that are given by finite linear combinations of characteristic functions for Lebesgue measurable sets, i.e. functions of the form

$$\Sigma_{n=1}^{N} \lambda_n \chi_{E_n}$$

where $\chi_E(t) = 1$ if $t \in E$ and $= 0$ if $t \notin E$. We assume that $\lambda_i \neq \lambda_j$ for $i \neq j$. Check for yourself that the simple functions are Lebesgue measurable. The key property for the simple functions is the following observation.

Theorem 1.2 (Approximation). *Let $f : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. Then there exists a sequence of simple functions ϕ_n, $n = 1, 2, \ldots$ such that*

1. $0 \leq \phi_1 \leq \phi_2 \leq \phi_3 \leq \ldots$
2. $\lim_{n\to\infty} \phi(t) = f(t)$ *for all* $t \in \mathbb{R}^n$
3. ϕ_n *converges uniformly to* f *on each set* $A \subset \mathbb{R}^n$ *where* f *is bounded.*

We note that the limit function for an increasing sequence of simple functions is also Lebesgue measurable. But also converse, i.e. that every Lebesgue measurable function (bounded below) can be obtained as the limit function for an increasing sequence of simple functions.

The proof for the theorem is quite simple. Set

$$\phi_n = \Sigma_{k=0}^{2^{2n}-1} k 2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n},$$

where

$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$$

and

$$F_n = f^{-1}((2^n, \infty]).$$

for $n = 1, 2, \ldots$. For an f of your choice draw the graphs for ϕ_n!

Next we introduce the term **almost everywhere**, abbreviated *a.e.*, which means everywhere except on a set of measure 0. To say that the functions f and g are equal *a.e.* means that the set where the functions differ must not be empty but have the Lebesgue measure 0. In the same way $f_n \to f$ pointwise *a.e.* means that the set where we do not have convergence is a 0–set. Since every subset of a 0–set is a 0–set we get

1. f Lebesgue measurable and $f = g$ *a.e.* implies that g is Lebesgue measurable.
2. f_n, $n = 1, 2, \ldots$, Lebesgue measurable and $f_n \to f$ pointwise *a.e.* implies that f is Lebesgue measurable.

Finally $f : \mathbb{R}^n \to \mathbb{C}$ is called Lebesgue measurable if both $\mathrm{Re}f$ and $\mathrm{Im}f$ are Lebesgue measurable. This is the same as saying that $f^{-1}(U) \in \mathcal{L}$ for every open set U in $\mathbb{C}$.

1.4 Integrals and convergence theorems

A complex-valued function f can uniquely be written as a sum of its real- and imaginary part

$$f = \mathrm{Re}f + i\,\mathrm{Im}f,$$

where $\mathrm{Re}f$ and $\mathrm{Im}f$ are real-valued. Both these functions can be written as a sum of the positive and the negative part of f. If f is real-valued we denote

$$f^+ = \max(f, 0)$$

and

$$f^- = \max(-f, 0).$$

Hence we get $f = f^+ - f^-$ (and $|f| = f^+ + f^-$). Since we want the integral operator

$$f \mapsto \int f\,dm$$

(not yet defined) to be linear on Lebesgue integrable functions we must have

$$\int f\,dm = \int (\mathrm{Re}f)^+\,dm - \int (\mathrm{Re}f)^-\,dm + i(\int (\mathrm{Im}f)^+\,dm - \int (\mathrm{Re}f)^-\,dm).$$

So it is enough to define

$$\int f\,dm$$

for all Lebesgue measurable functions $f : \mathbb{R}^n \to [0, \infty]$. This will be done in two steps.

Step 1 For $f = \Sigma_{n=1}^N \lambda_n \chi_{E_n}$, i.e. for a simple function f, we set

$$\int f\,dm = \Sigma_{n=1}^N \lambda_n m(E_n).$$

Step 2 If f is a Lebesgue measurable function we set

$$\int f\,dm = \sup\{\int \phi\,dm : \phi \text{ simple function, } 0 \leq \phi \leq f\}.$$

It can quite easily be shown that the integral is well-defined. The integral can attain the value $+\infty$ since we have not assumed any size condition for f. We let L^+ denote the set of all real Lebesgue measurable functions that takes values in $[0, \infty]$.

From the definition it follows that $f, g \in L^+$ and $f \leq g$ implies

$$\int f\, dm \leq \int g\, dm.$$

Moreover we let L^1 denote the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$\max(\int \mathrm{Re} f^+\, dm, \int \mathrm{Re} f^-\, dm, \int \mathrm{Im} f^+\, dm, \int \mathrm{Im} f^-\, dm) < \infty.$$

This is equivalent to

$$\int |f|\, dm < \infty.$$

Moreover we note that

$$|\int f\, dm| \leq \int |f|\, dm.$$

Finally we define

$$\int_E f\, dm = \int f\chi_E\, dm,$$

for E a Lebesgue measurable set in $\mathbb{R}^n$.

The question is then: What is the difference between the **Lebesgue integral** and the **Riemann integral**? The answer sits in the definitions. Let us for the moment assume that f attains its values in $[0, M]$ for some $M > 0$. Remember that the definition of the Riemann integral is based on splitting the x–axis into a union of tiny disjoint intervals I_k. Set $M_k = \sup_{I_k} f$ and $m_k = \inf_{I_k} f$. We get

$$\int f\, dx \approx \Sigma_k M_k |I_k|$$

provided $\Sigma_k M_k |I_k| \approx \Sigma_k m_k |I_k|$, where $|I_k|$ denotes the length of the interval I_k. With the notation $\approx$ we mean that the difference tends to 0 as $\sup_k |I_k| \to 0$. To have this we need f to be almost constant on every interval I_k (i.e. $M_k - m_k \approx 0$) or that the number of all intervals for which this is not true (i.e. $M_k - m_k \not\approx 0$) is small. Another way to phrase it is that f should be continuous except for a small set of points with Lebesgue measure 0. Using our special lingo we say that f is Riemann integrable if the set where f is discontinuous is a set of Lebesgue measure 0. In the previous example with $f = \chi_{\mathbb{Q} \bigcap [0,1]}$ the set of points of discontinuity is the whole interval $[0, 1]$ which has Lebesgue measure equal to 1 and not 0.

The definition of the Lebesgue integral is based on splitting the y–axis into small intervals $I_n^k = [k2^{-n}, (k+1)2^{-n})$. Here n indicates how fine the decomposition is, more precisely 2^{-n} is the length of the intervals. Comparing with the definition for simple functions (we assume that f is non-negative and $M < 2^n$) we have

$$\int f\, dm \approx \Sigma_k k 2^{-n} m(E_n^k),$$

where we observe that $|f - \phi_n| \le 2^{-n}$ on E_n^k. See page 6. For the sum to have a meaning it is needed that $m(E_n^k)$ and $m(F_n)$ are well-defined, which is guaranteed by the assumption that f is Lebesgue measurable. If we return to the function $f = \chi_{\mathbb{Q}\bigcap[0,1]}$ we see that f is 0 except at the rational points in the interval $[0,1]$, which is equal to $\{r_1, r_2, \ldots\}$. But every set $\{r_n\}$ is Lebesgue measurable with the measure 0 and a countable union of 0–sets is a 0–set. Hence we have $\int f\,dm = 0$.

We observe that every continuous function which is different from 0 only in a compact subset of $\mathbb{R}^n$ is Riemann integrable, that each Riemann integrable function (with finite $\|\cdot\|_1$–norm) is Lebesgue integrable **and**

$$\int_{\mathbb{R}^n} f(x)\,dx = \int f\,dm.$$

Here the LHS denotes the Riemann integral for f and the RHS denotes the Lebesgue integral for f.

Below we list some theorems that will become important to us for applications. It is important to note that the Lebesgue integral is an extension for the Riemann integral with the properties we wanted: powerful convergence theorems and the function space $(L^1, \|\cdot\|_1)$ is complete.

Theorem 1.3 (Lebesgue's monotone convergence theorem). *Let $(f_n)_{n=1}^\infty \subset L^+$ be a monotone increasing sequence of functions. Then we have*

$$\lim_{n\to\infty} \int f_n\,dm = \int \lim_{n\to\infty} f_n\,dm.$$

Theorem 1.4 (Fatou's lemma). *Let $(f_n)_{n=1}^\infty \subset L^+$ be a sequence of functions. Then we have*

$$\int \liminf_{n\to\infty} f_n\,dm \le \liminf_{n\to\infty} \int f_n\,dm.$$

Theorem 1.5 (Lebesgue's dominated convergence theorem). *Assume that $(f_n)_{n=1}^\infty$ is a sequence of complex-valued Lebesgue measurable functions such that $\lim_{n\to\infty} f_n = f$ a.e. Moreover assume that there exists a Lebesgue measurable function g such that*

$$|f_n| \le g \in L^1 \quad \text{all } n.$$

Then we have

$$f \in L^1$$

and

$$\lim_{n\to\infty} \int f_n\,dm = \int f\,dm.$$

Theorem 1.6 (Differentiation under the integral sign). *Assume that $f(t,x) : \mathbb{R}^n \times [a,b] \to \mathbb{C}$ and that $f(\cdot, x) : \mathbb{R}^n \to \mathbb{C}$ is a L^1–function for each $x \in [a,b]$. Set $F(x) = \int f(t,x)\,dm(t)$.*

- *Assume that there exists a $g \in L^1$ such that*

$$|f(t,x)| \le g(t) \quad \text{all } t, x.$$

Then we have

$$\lim_{x \to x_0} F(x) = F(x_0)$$

provided

$$\lim_{x \to x_0} f(t, x) = f(t, x_0) \quad \text{all } t.$$

- *Assume that* $\dfrac{\partial f}{\partial x}$ *exists and that there is a* $g \in L^1$ *such that*

  $$\left|\frac{\partial f}{\partial x}(t, x)\right| \le g(t) \quad \text{all } t, x.$$

  Then F *is differentiable and*

  $$F'(x) = \int \frac{\partial f}{\partial x}(t, x)\, dm(t).$$

We now assume that $n = 1$ and recall that a real function is continuously differentiable iff

$$f(x) = \int_a^x g(t)\, dt$$

where g is a continuous real function. Furthermore we have $f' = g$. What can be said about the function

$$\int_a^x g(t)\, dt$$

where $g \in L^1$?

To answer this question we introduce the concept of **absolutely continuous function**. We say that the real function f is absolutely continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\Sigma |b_n - a_n| < \delta$$

implies that

$$\Sigma |f(b_n) - f(a_n)| < \epsilon.$$

In particular this means that f is continuous and moreover uniformly continuous on the set where it is defined. $\Sigma \ldots$ stands for the sum for a finite series.

Theorem 1.7. *A real function* $f(x)$ *is given by* $\int_a^x g\, dm$*, where* g *is a locally*[5] *Lebesgue integrable function, iff* f *is absolutely continuous. In that case we have* $f' = g$ *a.e..*

From calculus course we remember that multiple Riemann integrals can be calculated by repeated integration. Is this still true for multiple Lebesgue integrals? The answer is contained in the following result.

Theorem 1.8 (Fubini–Tonelli's theorem). *Assume that* $f(\cdot, \cdot)$ *is Lebesgue measurable and that one of the following conditions are satisfied:*

[5] L^1_{loc} is defined below.

1. *(Tonelli)* $f \geq 0$

2. *(Fubini) one of the integrals* $\int |f(x,y)|\, dm(x,y)$, $\int (\int |f(x,y)|\, dm(y))\, dm(x)$, $\int (\int |f(x,y)|\, dm(x))\, dm(y)$ *is finite.*

Then the functions $f(\cdot, y)$, $f(x, \cdot)$, $\int f(\cdot, y)\, dm(y)$ *and* $\int f(x, \cdot)\, dm(x)$ *are Lebesgue measurable and*

$$\int f(x,y)\, dm(x,y) = \int (\int f(x,y)\, dm(y))\, dm(x) = \int (\int f(x,y)\, dm(x))\, dm(y).$$

1.5 L^p–spaces, Hölder's and Young's inequalities

For Lebesgue measurable functions f we define

$$\|f\|_p = (\int |f|^p\, dm)^{1/p}, \quad p \in [1, \infty)$$

and

$$\|f\|_\infty = \operatorname{ess\,sup} |f|.$$

Here ess sup for real-valued non-negative functions f denotes the quantity

$$\operatorname{ess\,sup} f = \inf\{k : k \geq f \text{ a.e.}\}.$$

We now define the L^p–space as the set of all Lebesgue measurable functions such that $\|f\|_p < \infty$. This is valid for $1 \leq p \leq \infty$ and we see that

- $\|f - g\|_p = 0$ iff $f = g$ *a.e.* Functions in L^p are identified if they are equal *a.e.*
- $f \in L^p$ implies that $|f| < \infty$ *a.e.*
- $f \in L^\infty$ and f continuous implies that $\|f\|_\infty = \sup |f|$. If f is not continuous then we obtain that the set of all x where $f(x) > \|f\|_\infty$ is a 0–set.

We claim that $\|\cdot\|_p$ really defines a norm. If $p = 1, \infty$ this is trivial. For $p \in (1, \infty)$ it is a consequence of **Hölder's inequality**

$$\|fg\|_1 \leq \|f\|_p\, \|g\|_q,$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p, q > 1$. This is established with a similar technique to that which was used for proving the corresponding statement for the sequence space l^p. This yields the **Minkowski's inequality**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

for $p \in (1, \infty)$.

We have now defined L^p as a normed space. The notations L^p_{loc} denotes the set of all Lebesgue measurable functions f for which $f\chi_E \in L^p$ for all compact Lebesgue measurable sets E in $\mathbb{R}^n$.

Theorem 1.9. *L^p with the norm $\|\cdot\|_p$ is a Banach space for $p \in [1, \infty]$. It is separable (there exists a countable dense set) for $p \in [1, \infty)$. If $(f_n)_{n=1}^\infty$ is a Cauchy sequence in L^p for $p \in [1, \infty)$ there exists a subsequence $(f_{n_k})_{k=1}^\infty$ that converges pointwise a.e.*

Try to prove this!!

Let f be a complex-valued function on $\mathbb{R}^n$. The closure of the set $\{x : f(x) \neq 0\}$ is called the **support** for f and we let C_0^∞ denote the set of all infinitely continuously differentiable functions with compact support.

Theorem 1.10. *For $p \in [1, \infty)$ we have*

1. $L^p \bigcap \{$ *simple functions* $\}$
2. C_0^∞

are both dense in L^p.

Finally we give some inequalities that can come in handy in many calculations.

Theorem 1.11 (Young's inequality). *Assume that $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable and that*

$$\max(\sup_x \int |k(x,y)|\, dm(y), \sup_y \int |k(x,y)|\, dm(x)) = M < \infty.$$

If $f \in L^p$ for some $p \in [1, \infty]$ then

$$F(x) = \int k(x,y) f(y)\, dm(y)$$

belongs to L^p and

$$\|F\|_p \leq M \|f\|_p.$$

Theorem 1.12 (Chebyshev's inequality). *Let $f \in L^p$, $p \in [1, \infty)$ and $\alpha > 0$ be given. Then we have*

$$m(\{x : |f(x)| > \alpha\}) \leq (\frac{\|f\|_p}{\alpha})^p.$$

In hindsight we note that functions that are Lebesgue integrable can be very wild but at the same time there are continuous nice functions that are close to the wild beasts in L^p-norm. Often in applications we want to prove that a certain function, appearing as a solution to some say integral equation, is continuous but from the first consideration we just obtain it as an element in L^p. However the continuity property for the function can then be established from the specific problem. What the L^p–theory has contributed with is the existence of a function that can be proven to have some good properties.

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